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The Second Quantized Approach to the Study of Model Hamiltonians in Quantum Hall Regime

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by

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ABSTRACT OF THE DISSERTATION

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Professor Alexander Seidel, Chair

The study of model Hamiltonians, whose exact ground states are well known quantum Hall states, have been investigated solely in the literature from the point of view of first quantized approach in the last few decades. However, a second quantized approach to such Hamiltonians has not been taken in the same measure. In this dissertation, we study Haldane pseudopotential, which is the parent Hamiltonian of $1/M$ Laughlin state, and Trugman-Kivelson pseudopotential, which is the parent Hamiltonian of $2/5$ unprojected Jain state. We find that the study of these Hamiltonians in their second quantized forms not only reproduces all properties of their zero modes already known in first quantized approach, but also sheds lights on general studies of such structures of these Hamiltonians as frustration free lattice Hamiltonians.

Chapter 1

Introduction to Quantum Hall Effect

In this chapter, a review of the history of study of quantum Hall effect will be presented. The emphasis will be on the experimental facts and their theoretical explanations.

1.1 The classical Hall effect

In 1879, a graduate student, Edwin Hall, at Johns Hopkins University devised an experiment to test the hypothesis that "If the current of electricity in a fixed conductor is itself attracted by a magnet, the current should be drawn to one side of the wire, and therefore the resistance experienced should be increased."[34]. He discovered that a voltage is built up across the wire when the wire is under a magnetic field at right angle to the direction of the current through the wire, an effect named after him since then. In modern language, the reason for the observed phenomenon is that the carriers in the wire undergo deflection due to the Lorentz force and hence accumulate at the edges. Therefore, a voltage is built up across the wire. As long as the electric force due to this voltage counterbalances the Lorentz force, the accumulation of carries

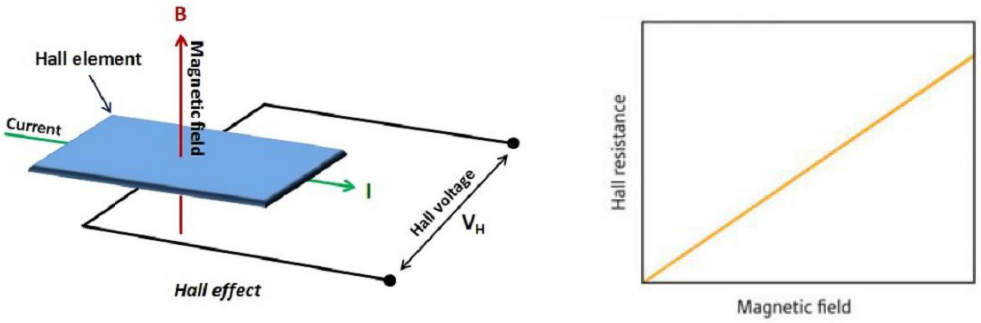

Figure 1.1: Schematics of classical Hall effect. The Hall element is in a magnetic field which is at right angle to the direction of the current through the element. A Hall voltage is found to exist in the direction perpendicular to both the current and the magnetic field. It is found out that the Hall resistance defined as the ratio of the Hall voltage V_H to the current I is proportional to the magnetic field.

at two edges stops and the voltage remain stable. The Hall resistance R_H, defined as the ratio of the voltage V_H to the current I, is proportional to the magnetic field B and inversely proportional to the carrier density n. If the type of charges carried by the carriers in the wire changes the sign, the voltage will also change its direction accordingly. A sketch of the experimental setup and finding is presented in Fig. 1.1. The discovery of the classical Hall effect has since then been widely used to determine the type and density of carriers in semiconductors and also been used to measure the magnetic field.

1.2 The integer quantum Hall effect

Nearly one hundred years after Edwin Hall discovered the Hall effect, a physicist whose name is Klaus von Klitzing, when investigating the electronic properties of a 2D(2 dimensional) electron gas in the inversion layer of a MOSFET(metal-oxide-semiconductor field-effect-transistor), found that the Hall resistance becomes quantized as several plateaus whose values are independent of the magnetic field when the sample is in a magnetic field as high as 15T with ambient temperature under 3K[44].

What is amazing is that at these plateaus, the Hall resistance is quantized nearly exactly as the Planck's constant h divided by an integer ν and the square of the electric charge e:

$$R_H = \frac{h}{\nu e^2}, \tag{1.1}$$

with the error being less than one part in one hundred million. Moreover, the longitudinal resistance nearly vanishes when the Hall resistance is quantized while being at a peak when the Hall resistance goes from one plateau to the adjacent plateau. The nearly exact quantization of Hall resistance has been used ever since then in the standard of the resistance in terms of fundamental constant h and e. The setup of the experimental apparatus and the results are shown in Fig. 1.2.

The vanishing of the longitudinal resistance and quantization of Hall resistance can be easily understood in the context of quantum mechanics of a single electron in a magnetic field. As is well known in physics, the kinetic energy of a single electron in 2D under a magnetic field splits into discrete values known as Landau levels, each one of which has a degeneracy equal to the number of flux quanta piercing the sample[1]. When the Fermi energy is between two neighboring Landau levels, electrons cannot be scattered into higher Landau levels at low temperature as the Landau level separation $\hbar\omega$ is much larger than $k_B T$ where $\hbar$ is the reduced Planck's constant $h/(2\pi)$, the cyclotron frequency ω is equal to eB/m where B is the magnetic field and m is the effective mass of an electron, and k_B is the Boltzmann constant. According to the

[1]See Appendix A for details.

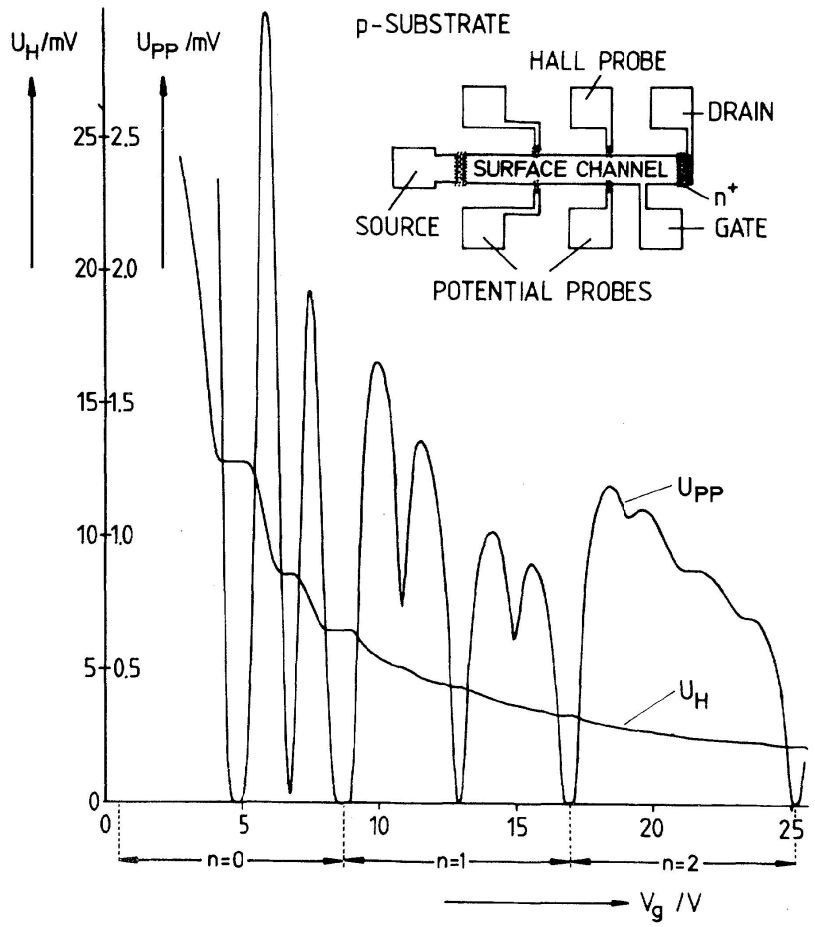

Figure 1.2: The Hall voltage and the longitudinal voltage as functions of the gate voltage at $T = 1.5K$ and $B = 18T$. The electron density n in filling factor $\nu = \frac{nh}{eB}$ is controlled by the gate voltage. As shown in the figure, the Hall voltage becomes quantized at several plateaus while the longitudinal voltage is zero. Between adjacent zeros, the longitudinal voltage exhibits peaks while the Hall voltage is between two neighboring plateaus. Copyright © the American Physical Society. The right to use this figure from [44] has been granted by the American Physical Society.

Arrehenius formula, the longitudinal resistance is given by:

$$R_L = R\ e^{-\frac{\Delta}{k_B T}}, \tag{1.2}$$

where the excitation gap Δ is just the Landau level separation $\hbar\omega$ in the current case. Therefore, the longitudinal resistance approaches zero in this region.

It turns out that the quantization of the Hall resistance is insusceptible to the geometry of the sample, the weak disorder and electron concentration. The accuracy of the quantization of the Hall resistance derives from the topological nature of the system. Let us write down the conductivity tensor in terms of the resistivity tensor:

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \frac{h}{\nu e^2} \\ -\frac{h}{\nu e^2} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\frac{\nu e^2}{h} \\ \frac{\nu e^2}{h} & 0 \end{pmatrix} \tag{1.3}$$

It is found out that the Hall conductivity σ_{xy} is proportional to the sum of Chern numbers of Landau levels below the Fermi energy[96, 45]:

$$\sigma_{xy} = -\frac{e^2}{h} \sum_{\varepsilon_\alpha < E_F} C_\alpha, \tag{1.4}$$

where C_α is the Chern number of the band with the index α whose energy ε_α is below the Fermi energy E_F. The Chern number of a band is defined as the integral of Berry flux associated with that band in the magnetic Brillouin zone:

$$C = \frac{1}{2\pi i} \iint_{BZ} (\partial_1 A_2(\mathbf{k}) - \partial_2 A_1(\mathbf{k})) d^2k \tag{1.5}$$

with Berry connection

$$A_\mu(\mathbf{k}) = \langle n_\mathbf{k} | \partial_{k_\mu} | n_\mathbf{k} \rangle \tag{1.6}$$

in terms of normalized wave function $|n_\mathbf{k}\rangle$ of that band.

It can be proved that the Chern number vanishes if the Berry connection is smooth and well defined throughout the whole magnetic Brillouin zone. The nonzero Chern number is entirely determined by the topology of the Berry connection $A_\mu(\mathbf{k})$. When-

ever we cannot define a unique and smooth Berry connection in the magnetic Brillouin zone, we have to divide the magnetic Brillouin zone into several parts, inside each part the Berry connection is smooth and well defined. For simplicity, we will consider the case that there are two such parts with a contour c as the common boundary. The wave function and the Berry connections in each part of the magnetic Brillouin zone are connected by a gauge transformation:

$$\begin{aligned} |n'_{\mathbf{k}}\rangle &= e^{i\lambda(\mathbf{k})}|n_{\mathbf{k}}\rangle, \\ A'_{\mu}(\mathbf{k}) &= A_{\mu}(\mathbf{k}) + i\partial_{k_\mu}\lambda(\mathbf{k}). \end{aligned} \tag{1.7}$$

Due to the definition in Eq. 1.5, the Chern number is just the winding number of $\lambda(\mathbf{k})$ along the path of the contour c, which is a nonzero integer:

$$C = \frac{1}{2\pi}\int_{c} \nabla_{\mathbf{k}}\lambda(\mathbf{k}) \cdot d\mathbf{k}. \tag{1.8}$$

The above argument can easily be generalized to the case in which the magnetic Brillouin zone needs to be divided into more than two parts such that the Berry connection is smooth and well defined individually in each part. Thus the Hall conductivity σ_{xy} is quantized as $\frac{e^2}{h}$ times an integer by virtue of Eq. (1.4). The association of the Hall conductivity σ_{xy} with the topological Chern number clarifies why it is insensitive to the geometry and the electron concentration of the sample.

When there are disorders in the sample due to donor impurities, the Landau levels will be broadened to have nonzero energy widths as shown in Fig. 1.3. The extended state in the middle of each broadened Landau level is separated from localized states in the tails of the band by two mobility edges. As long as disorders are weak, there will be a nonzero mobility gap between two neighboring extended states. When the Fermi energy is inside the mobility gap, the Hall conductivity σ_{xy} will remain quantized since the localized states do not contribute to the Hall conductivity [64, 2, 95, 35]. Only when the Fermi energy is inside the range of extended states does the Hall conductivity go from one plateau to the adjacent plateau.

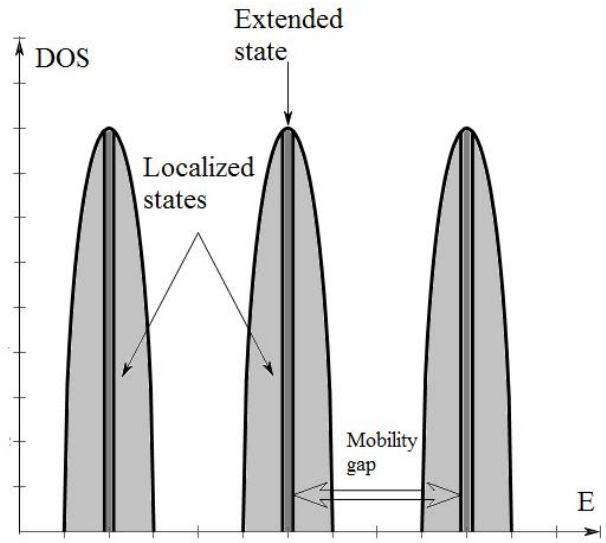

Figure 1.3: The density of states of broadened Landau levels versus the energy. In the presence of disorders in the sample, the Landau levels will be broadened. The middle of each broadened Landau level is the extended state. In the tails of the band are localized states.

1.3 The fractional quantum Hall effect

Just in two years after the discovery of the integer quantum Hall effect, a phenomenon known as the fractional quantum Hall effect was discovered by Daniel Tsui and Horst Störmer [99]. It was found that in GaAs-AlGaAs heterojunctions, the Hall resistance R_H can also be quantized as $h/(\nu e^2)$ where $\nu = 1/3$. This quantization of the Hall resistance at fractional filling factors was initially attributed to the formation of the Wigner crystal or charge density wave. Later the quantization of the Hall resistance at other fractional filling than 1/3 was also discovered [88, 19, 109, 40]. See Fig. 1.4 for details.

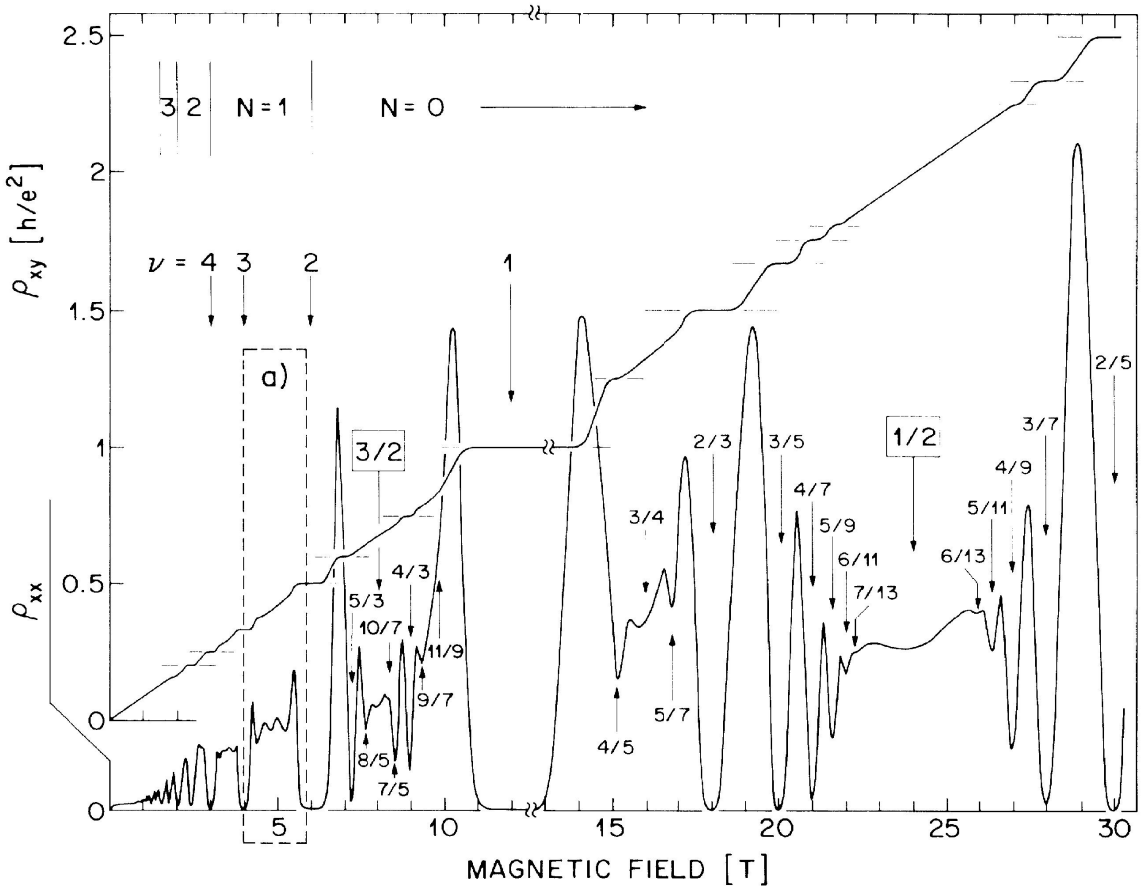

Figure 1.4: The Hall resistivity and the longitudinal resistivity as functions of the magnetic field. In the plateau region, the diagonal resistivity ρ_{xx} vanishes or dips whereas the Hall resistivity ρ_{xy} is quantized as $\rho_{xy} = \frac{h}{\nu e^2}$ where filling factor ν is defined as $\frac{nh}{eB}$. The quantization of ρ_{xy} involves both integer filling factor $\nu = 1, 2, 3, 4$ and 5 and a series of fractional filling factors in which $\nu = \frac{p}{q}$ with p and q being co-prime integers. The fractional quantum Hall effect is prominent at filling factors $\frac{2}{5}$, $\frac{3}{7}$, $\frac{4}{9}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{7}$, $\frac{5}{9}$ and $\frac{4}{3}$. The quantization of ρ_{xy} at 1/3 is not shown in the figure. Copyright © the American Physical Society. The right to use this figure from [109] has been granted by the American Physical Society.

1.4 Theoretical explanations of the fractional quantum Hall effect

The fractional quantum Hall effect is puzzling since the quantization of Hall resistivity appears at fractional filling factors when a certain Landau level is partially occupied. In this case, the kinetic energies of electrons are quenched so that interactions among electrons have to be considered. This is due to the fact that the Landau level separation is linear in the magnetic field while the Coulomb repulsion is proportional to the square root of the magnetic field[38]:

$$\hbar\omega \sim 20B[T]\ K, \tag{1.9}$$

$$\frac{e^2}{\epsilon l_B} \sim 50\sqrt{B[T]}\ K. \tag{1.10}$$

Therefore, when the many-body Hamiltonian of the system

$$H = \sum_i \frac{\mathbf{\Pi}_i^2}{2m} + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) \tag{1.11}$$

is projected to the fractionally filled Landau level, the kinetic energies of electrons can be dropped out, leaving only Coulomb repulsions in the Hamiltonian. This Hamiltonian, by its intrinsic many-body nature, has no small parameters to begin with, rendering traditional perturbation theory approach useless. To circumvent this obstacle, R. Tao and Yong-shi Wu[94] gave an explanation of fractional quantum Hall effect based on a gedanken experiment proposed by Robert Laughlin to explain the integer quantum Hall effect using gauge invariance[48]. Later on, this proof was formalized by R. Tao and F. D. M. Haldane[92]. In the next two sections, We would introduce the Laughlin's gedanken experiment and the proof of R. Tao and F. D. M. Haldane.

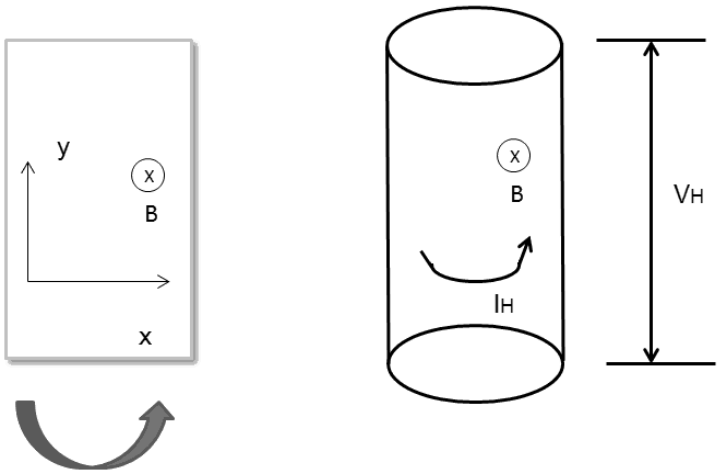

Figure 1.5: The 2D sample as well as the cylinder formed by wrapping the sample in the x direction.

1.5 Laughlin's gedanken experiment

Suppose we have a 2D sample in a perpendicular magnetic field as in Fig. 1.5. We can wrap the sample in the x direction to get a cylinder. Suppose we change the magnetic flux piercing the cylinder Φ by one elementary flux quantum $\Phi_0 = h/e$, then by gauge invariance the system maps back to itself. The net change to the system is that N electrons are transferred from the bottom to the top of the cylinder. The energy change to the system is therefore $\Delta\epsilon = NeV_H$ where V_H is the Hall voltage. The current in the x direction is

$$I_H = \frac{\Delta\epsilon}{\Delta\Phi} = \frac{Ne^2V_H}{h}. \tag{1.12}$$

Thus the Hall conductivity

$$\sigma = \frac{I_H}{V_H} = \frac{Ne^2}{h}. \tag{1.13}$$

Tao and Wu further extended Laughlin's argument to fractional quantum Hall effect, arguing that it may require $q\Phi_0$ magnetic flux to map the system back to itself and p electrons are transferred in this process. Therefore the Hall conductivity

$$\sigma = \frac{I_H}{V_H} = \frac{p}{q}\frac{e^2}{h}, \tag{1.14}$$

accounting for the experimentally observed facts.

These arguments are successful in explaining the fractional quantization of the Hall conductivity, but it is not clear why a certain number of electrons are transferred during the process in which the magnetic flux is changed. In the next section, we will show the rigorous proof of fractional quantization by Tao and Haldane.

1.6 Tao and Haldane's proof of the fractional quantization of Hall conductivity

According to Tao and Haldane, the Hamiltonian of N electrons in a 2D rectangle of length L_x and of width L_y in a magnetic field

$$H = \frac{1}{2m}\sum_i \left((\Pi_{i,x} + \frac{\hbar\alpha_x}{L_x})^2 + (\Pi_{i,y} + \frac{\hbar\alpha_y}{L_y})^2 \right) + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j), \tag{1.15}$$

where α_x and α_y are two magnetic fluxes threading the torus formed by the rectangle, can be written as the sum of a center-of-mass coordinate part and a relative coordinate part: $H = H_c + H_r$ with

$$H_c = \frac{1}{2mN}\left((\Pi_x^c + \frac{N\hbar\alpha_x}{L_x})^2 + (\Pi_y^c + \frac{N\hbar\alpha_y}{L_y})^2 \right) \tag{1.16}$$

and

$$H_r = \frac{1}{2mN}\sum_{i<j} (\mathbf{\Pi}_i - \mathbf{\Pi}_j)^2 + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j). \tag{1.17}$$

Dynamical momenta operators are $\Pi_x = p_x + eA_x$ and $\Pi_y = p_y + eA_y$ with Landau gauge $\mathbf{A} = B(0, x, 0)$. Here Π^c is the sum of all Π. We can further define pseudomomentum operators:

$$\kappa_x = -i\hbar\frac{\partial}{\partial x} + eBy \tag{1.18}$$

and

$$\kappa_y = -i\hbar\frac{\partial}{\partial y} \tag{1.19}$$

satisfying $[\mathbf{\Pi}, \kappa] = 0$. When there is no magnetic field, the Hamiltonian on the torus formed from the rectangle commutes with the ordinary translation operators. However, in the presence of a magnetic field, it fails to do so. In this case, we can define magnetic translation operators:

$$T_1 = e^{\frac{iL_x\kappa_x^c}{\hbar N_\Phi}} \tag{1.20}$$

and

$$T_2 = e^{\frac{iL_y\kappa_y^c}{\hbar N_\Phi}} \tag{1.21}$$

where N_Φ is the number of magnetic flux quanta $BL_xL_y/(h/e)$ which is chosen to be an integer and κ^c is the sum of all κ.

T_1 and T_2 all commute with H. However, T_1 does not commute with T_2:

$$T_2T_1 = T_1T_2e^{i2\pi\frac{p}{q}} \tag{1.22}$$

where $p/q = N/N_\Phi$, which is just the filling factor of Landau levels. As a result, we can only diagonalize H, T_1^q and T_2 simultaneously since T_1^q commutes with T_2:

$$T_2T_1^q = T_1^qT_2e^{i2\pi p} = T_1^qT_2. \tag{1.23}$$

We can take the simultaneous eigenstate of H, T_1^q and T_2 to be ψ_0 and take the eigenvalues of T_1^q and T_2 corresponding to ψ_0 as $e^{iq\phi_1}$ and $e^{i\phi_2}$, respectively.

We can find that the ground states of H on this torus has a q-fold degeneracy: $\psi_0, T_1\psi_0, T_1^2\psi_0, T_1^3\psi_0, \ldots, T_1^{q-1}\psi_0$ all have the same eigenenergy. As H_c and H_r com-

mute, we can write the q-fold ground state of H as $\psi_k = \psi_{c,k}\psi_r \;\; (k = 0, 1, \ldots, q-1)$ where $\psi_k = T_1^k \psi_k$. Then it turns out that $\psi_{c,k}$ is also the eigenstate of T_1^q and T_2:

$$T_1^q \psi_{c,k} = T_1^q \frac{\psi_k}{\psi_r} = T_1^q \frac{T_1^k \psi_0}{\psi_r} = \frac{T_1^k T_1^q \psi_0}{\psi_r} = \frac{T_1^k e^{iq\phi_1}\psi_0}{\psi_r} = e^{iq\phi_1}\psi_{c,k}, \tag{1.24}$$

$$T_2 \psi_{c,k} = T_2 \frac{\psi_k}{\psi_r} = T_2 \frac{T_1^k \psi_0}{\psi_r} = \frac{T_1^k T_2 e^{i2\pi ps/q}\psi_0}{\psi_r} = \frac{T_1^k e^{i2\pi ps/q} e^{i\phi_2}\psi_0}{\psi_r} = e^{i(2\pi ps/q+\phi_2)}\psi_{c,k}, \tag{1.25}$$

Now assume the initial state of the system is a superposition of all q-fold ground state:

$$\Psi = \sum_{k=1}^{q} c_k \psi_k. \tag{1.26}$$

Then we can find that if we increase α_x by 2π, the Ψ will not go back to itself. On the contrary, we need to increase α_x by at least $2q\pi$ to move the system back to itself:

$$\Psi(\alpha_x + 2q\pi, \alpha_y) = \Psi(\alpha_x, \alpha_y) e^{-i2q\pi \sum\limits_i x_i/L_x - ip\alpha_y - iq\phi_2}. \tag{1.27}$$

Similarly, we have:

$$\Psi(\alpha_x, \alpha_y + 2q\pi) = \Psi(\alpha_x, \alpha_y) e^{-i2q\pi \sum\limits_i y_i/L_y + iq\phi_1}. \tag{1.28}$$

Using the Kubo formula, we finally have the fractionally quantized Hall conductivity

$$\begin{aligned} \sigma =& \frac{ie^2}{\hbar(2q\pi)^2} \iint_{\substack{0\leqslant\alpha_x\leqslant 2q\pi \\ 0\leqslant\alpha_y\leqslant 2q\pi}} \left(\frac{\partial j_y}{\partial \alpha_x} - \frac{\partial j_x}{\partial \alpha_y} \right) \\ =& \frac{p}{q}\frac{e^2}{h}, \end{aligned} \tag{1.29}$$

where p/q is equal to the filling factor N/N_Φ.

Tao and Haldane also showed that if there are weak impurities present, the form of Hall conductivity is unchanged if the characteristic energy of weak impurities is still less than the gap induced by the H_r. X. G. Wen and Q. Niu also argued that the q^g-fold degeneracy arising from the center of mass part of Hamiltonian H_c embedded in a Riemann surface of genus number g can be used to characterize the topological

order of the system[105].

We must mention that although the above theory explains very well the quantization of Hall conductivity at fractional filling factors, it cannot give the condition under which the system has a gap at certain filling factors observed in experiments such as $\nu = 1/3$ and $2/5$. The existence of a gap in the thermodynamic limit at experimentally observed filling factors is indeed an challenging issue, part of which will be clarified in Section 1.10.

1.7 Laughlin's trial wave function

Motivated by the wave function with relative angular momentum m and total angular momentum M of two electrons in the lowest Landau levels

$$(z_1 - z_2)^m (z_1 + z_2)^{M-m} e^{-\frac{|z_1|^2+|z_2|^2}{4l_B^2}}, \tag{1.30}$$

Robert Laughlin proposed a wave function to describe the ground state of the many-body Hamiltonian (1.11) of N particles in the presence of a magnetic field in the disk geometry at filling factor $1/m$[49]:

$$\psi_{\frac{1}{m}} = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m e^{-\sum_{k=1}^{N} \frac{|z_k|^2}{4l_B^2}} \tag{1.31}$$

where $z = x + i\ y$ is the complex coordinate.

Laughlin gave several justifications for this wave function:

(1) It is an antisymmetric wave function whose polynomial part is holomorphic in $z_1, z_2, \ldots, z_N$, as befits a wave function of N electrons in the lowest Landau level.

(2) It is a homogeneous function of $z_1, z_2, \ldots, z_N$, with a total angular momentum $M(N-1)N/2$ since the total angular momentum of electrons on a disk in a magnetic field is conserved.

(3) It describes a uniform liquid with the correct filling factor $1/m$. To see the

uniformity of electronic density, the probability density $|\psi_{\frac{1}{m}}|^2$ can be written as a canonical ensemble distribution function of a two-dimensional one-component plasma:

$$|\psi_{\frac{1}{m}}|^2 = e^{-\beta\epsilon}, \tag{1.32}$$

with $\beta = 2/m$ and

$$\epsilon = -m^2 \sum_{1\leqslant i<j\leqslant N} \ln|z_i - z_j| + \frac{m}{4}\sum_{k=1}^{N}\frac{|z_k|^2}{l_B^2}. \tag{1.33}$$

The first term in ϵ is the potential energy of particles of charge m in 2D and the second term is the sum of potential energy of a particle at $|z|^2$ interacting with the back ground charge of density $\rho = -\frac{1}{2\pi l_B^2}$. Charge neutrality requires that the number density of electrons n obeys

$$n\ m + \rho = 0. \tag{1.34}$$

It follows that $n = 1/(2\pi l_B^2 m)$ which is constant. According to Eq. A.18, filling factor is just $\nu = 1/m$.

Laughlin calculated the overlaps of his trial wave functions of different relative angular momentum m with the ground state wave functions of the same relative angular momentum numerically computed from the 2D Hamiltonian of three particles interacting with Coulomb, logarithmic and Gaussian potential, respectively, in the presence of a perpendicular magnetic field. What he found is that all these overlaps are between 0.995 and 1, justifying the validity of his proposed wave function as a trial wave function.

A question can then be naturally asked when the Laughlin's wave function ceases to describe the ground state of the many-body Hamiltonian (1.11). It turns out the 2D one-component plasma in the plasma analogy of Laughlin's wave function undergoes a phase transition from a liquid to a crystal state at $m = 70$[18].

However, the Laughlin state fails to characterize the ground state of the 2D electron system even at higher electron number densities when the Laughlin state is unstable to the formation of the charge density wave (CDW) state. The CDW state

is found to have the same periodicity as that of classical Wigner crystal[114] but have large overlaps of wave functions of electrons at neighboring lattice sites[113]. The CDW state is pinned and does not have the overall drift velocity E/B, hence it does not exhibit the quantum Hall effect. Moreover, it was found that the energy of the CDW state is a smooth function of filling factor ν[114]. Therefore, the gap[42]

$$\Delta = \nu^2 \left(\left. \frac{\partial \epsilon(\nu)}{\partial \nu} \right|_+ - \left. \frac{\partial \epsilon(\nu)}{\partial \nu} \right|_- \right) \tag{1.35}$$

vanishes for the CDW state at every filling factor and thus does not favor any filling factor such as p/q. On the contrary, the ground state energy of 2D electron system is shown to be lower than the energy of the CDW state in the range $0.29 \leq \nu \leq 0.6$ and demonstrates cusps at $\nu = 1/3$ and $2/5$[113]. It then follows that the fractional quantum Hall effect at $\nu = 1/3$ and $2/5$ are protected by gaps and the CDW state is not the ground state when $0.29 \leq \nu \leq 0.6$. Further numeric study revealed that the Laughlin state transitions to the CDW state at a filling factor slightly over $1/7$[46]. Density matrix renormalization group(DMRG) studies of up to 25 electrons also confirmed the excitation gaps at $\nu = 1/3, 2/5$ of Laughlin state and the transition of the Laughlin state to the CDW state[86].

In conclusion, it is legitimate to use the Laughlin's wave function of filling factor $1/m$ with $m = 3, 5, 7$ to model the corresponding ground state of the many-body Hamiltonian (1.11).

1.8 Excitations of Laughlin state

Laughlin have also proposed wave functions for quasiholes and quasielectrons. Due to the fact that the quasihole wave function is less complicated than the quasielectron wave function and another fact that both Laughlin and quasihole wave functions are exact ground states of a model Hamiltonian which will be introduced in the next section, we will mention only the quasihole in this section.

For a quasihole located at η, the wave function is

$$\psi_{1 \text{ quasihole}} = \prod_{l=1}^{N} (z_l - \eta) \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m e^{-\sum_{k=1}^{N} \frac{|z_k|^2}{4l_B^2}} \tag{1.36}$$

The motivation for the form of this wave function is as follows: If a flux quantum is inserted at the origin, the number of allowed orbitals in the lowest Landau level would increase by 1, so the power of each coordinate should also increase by 1. Therefore we need to multiply the Laughlin wave function by a symmetric function $\prod_{l=1}^{N} z_l$. The net result is the creation of a quasihole at the origin since the probability of find electrons at the origin would be zero. Similarly, to get the wave function describing a quasihole at η, all we have to do is to multiply the Laughlin wave function by $\prod_{l=1}^{N} (z_l - \eta)$.

To intuitively see the effective charge of the quasihole at η, we can write down the wave function describing m quasiholes at η:

$$\psi_{m \text{ quasiholes}} = \prod_{l=1}^{N} (z_l - \eta)^m \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m e^{-\sum_{k=1}^{N} \frac{|z_k|^2}{4l_B^2}}. \tag{1.37}$$

Then we can add another electron with a wave function $\delta(z_{N+1} - \eta)\exp(-|z_{N+1}|^2/(4l_B^2))$ to this system of N electrons and m quasiholes. We can thus get the $N+1$-particle Laughlin wave function by multiplying the m-quasihole wave function by $N+1$-th electron wave function $\delta(z_{N+1} - \eta)\exp(-|z_{N+1}|^2/(4l_B^2))$ and integrating over η. To go from N electrons to $N+1$ electrons, the charge neutrality requires that the background charge increases by $+e$ which is carried by m quasiholes. Hence the effective charge of each quasihole is $+e/m$.

Theory has predicted that the shot noise power in tunneling experiments is directly proportional to the carrier charge, thus it is natural to probe the charge carried by the quasiholes by measuring the shot noise. The existence of fractionally charged quasiholes has been confirmed in those shot noise experiments[76, 26].

1.9 Parent Hamiltonian of Laughlin state

Although the general many-body Hamiltonian (1.11) seems to render an exact solution impossible, the $\nu = 1/M$ Laughlin state with the wave function $\prod_{i<j}(z_i - z_j)^M \exp(-\sum_k |z_k|^2/4l_B^2)$ can be shown to be an exact zero energy ground state of a model Hamiltonian[31]:

$$H = \sum_{\substack{0\leq m<M \\ (-1)^m=(-1)^M}} f_m V_m \tag{1.38}$$

where f_m is a positive coefficient and $V_m = \sum_{i<j} P_{ij}^m$ is called V_m Haldane pseudopotential. Here i and j are particle indices and the P_{ij}^m are two-body operators. Each P_{ij}^m projects a pair of particles with indices i and j onto a two-body relative angular momentum eigenstate $|m\rangle$ which has the eigenvalue m:

$$P_{ij}^m = |m\rangle\langle m|, \tag{1.39}$$

with

$$\langle \mathbf{r}|m\rangle = \frac{z_r^m e^{-|z_r|^2/8l_B^2}}{\sqrt{2\pi 2^{2m+1} l_B^{2m+2} m!}}, \tag{1.40}$$

where $z_r = z_i - z_j$.

The motivation for the representation of the Hamiltonian in terms of a sum of projection operators is the following: when projected onto the lowest Landau levels, the degree of freedom associated with the kinetic part is frozen out, leaving only the interaction term in the Hamiltonian. As the 2D system has a translational invariant symmetry, the interactions only depend on the degree of freedom associated with relative coordinates. Thus the projected Hamiltonian can be written as

$$\begin{aligned} H &= \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) \\ &= \sum_{i<j} \sum_{m,m'} |m\rangle\langle m|V(\mathbf{r}_i - \mathbf{r}_j)|m'\rangle\langle m'| \end{aligned}. \tag{1.41}$$

Since the 2D system also has a rotational invariant symmetry, the matrix of V in the basis of two-body relative angular momentum eigenstates would be diagonal. Therefore the above Hamiltonian can be simplified as

$$\begin{aligned} H &= \sum_{i<j} \sum_{m} |m\rangle\langle m|V(\mathbf{r}_i - \mathbf{r}_j)|m\rangle\langle m| \\ &= \sum_{i<j} \sum_{m} f_m P_{ij}^m \end{aligned} . \tag{1.42}$$

where $f_m = \langle m|V|m\rangle$. Here we have expanded the interaction Hamiltonian of electrons in 2D in terms of Haldane pseudopotentials. In fact, any translational and rotational invariant Hamiltonian can be expanded in the same manner.

The fact that the $\nu = 1/M$ Laughlin state is an exact zero energy ground state of the special Hamiltonian (1.38) can be easily seen from that any pair of two particles in the Laughlin $1/M$ state has a relative angular momentum M which is larger than m.

For the Coulomb potential, positive f_m is given by [54]

$$f_m = \frac{e^2}{l_B}\sqrt{\frac{\pi}{4}}\frac{(2m-1)!!}{2^m m!}. \tag{1.43}$$

As the Coulomb repulsion among electrons may be screened, the actual interaction may deviate from the pure Coulomb potential. As a result, the f_m's in the literature are often adjusted accordingly to study the properties of the ground state and the elementary excitations.

The quasihole state is also a zero energy ground state of the Hamiltonian (1.38) since any pair of particles in the quasihole state still has a relative angular momentum M, as easily seem from the quasihole wave function (1.36).

The Haldane pseudopotential has been investigated thoroughly in numerics, but a systematic study of zero energy ground states of Haldane pseudopotentials in a purely second quantized language is still lacking. In Chapter 2, we will motivate our purely second quantized approach to the study of zero energy ground states of Haldane pseudopotentials, investigate the algebraic properties of operators pertaining

to second quantized Haldane pseudopotentials, and consequently give a recursive formula for zero energy ground states based on these algebraic properties.

1.10 Composite fermions

In order to theoretically explain why the fractional quantum Hall effect prefers certain filling factors, Jainendra Jain proposed a theory relating the fractional quantum Hall effect of electrons to the integer quantum Hall effect of a new kind of particles[37]. His theory is that flux quanta of magnitude $2p\ \Phi_0$ parallel to the external magnetic field, where p is an integer and the elementary flux quantum Φ_0 is equal to h/e, will be attached to each electron, forming composite fermions. Jain further assumed that upon this attachment, strongly interacting electrons will become weakly interacting composite fermions.

It is natural to think that composite fermions will occupy Landau levels just like electrons do in a magnetic field. However, since each composite fermion comprises of an electron and $2p$ elementary flux quanta, the "Landau levels" in which composite fermions reside should originate from an effective magnetic field B^* which is different from the original magnetic field B. Jain thus called these modified Landau levels the "Λ levels" to distinguish them from real Landau levels. It is further argued by Jain that the fractional quantum Hall effect of electrons at $\nu = \frac{k}{2pk\pm1}$, where n is an integer is in one-to-one correspondence to the integer quantum Hall effect of composite fermions at $\nu^* = k$.

To appreciate this claim, we must determine the effective magnetic field in the composite fermion approach in the first place. The total magnetic flux threading the 2D sample is originally BA before the attachment of the flux quanta to each electron, where B is the original external magnetic field and A is the area of the 2D sample. After each electron absorbs flux quanta of $2p\Phi_0$, the remaining magnetic flux piercing the sample would be $BA - N2p\Phi_0$ with N being the number of electrons in the area A. This remaining magnetic flux is equal to the effective magnetic field B^* times the

area of the sample A. Thus we arrive at an identity

$$B^*A = BA - 2pN\Phi_0. \tag{1.44}$$

We can multiply both sides of Eq. 1.44 by $\frac{e}{nAh}$ and use $\nu = \frac{nh}{eB}$, and $\nu^* = \frac{nh}{e|B^*|}$ to get

$$\frac{1}{\nu^*} = \pm\left(\frac{1}{\nu} - 2p\right). \tag{1.45}$$

Therefore, we can finally obtain

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}. \tag{1.46}$$

If the number of Λ levels occupied by composite fermions is k, the filling factor of Landau levels occupied by electrons is $\nu = \frac{k}{2pk\pm1}$, accounting for most of the filling factors found in the fractional quantum Hall regime.

For example, if $p = 1$, ν would be $\frac{k}{2k\pm1}$. The series in which the minus sign is used gives $1, 2/3, 3/5, \ldots$ and the positive branch gives $1/3, 2/5, 3/7, \ldots$. The energy gap at $\frac{k}{2k\pm1}$ is due to the Λ level separation

$$\begin{aligned}\Delta =& \frac{\hbar e|B^*|}{m_{\mathrm{CF}}} \\ =& \frac{\hbar e}{m_{\mathrm{CF}}}\left(\pm(B - 2n\frac{h}{e})\right) \\ =& \pm\frac{\hbar e}{m_{\mathrm{CF}}}\left(B - B_{\nu=\frac{1}{2}}\right).\end{aligned} \tag{1.47}$$

Thus when $1/2 < \nu < 1$, the energy gap would be $\hbar e(B_{\nu=\frac{1}{2}} - B)/m_{\mathrm{CF}}$ which is linear in $B_{\nu=\frac{1}{2}} - B$. On the other hand, when $1/3 < \nu < 1/2$, the energy gap would be linear in $B - B_{\nu=\frac{1}{2}}$. Likewise, if $p = 2$ and $1/4 < \nu < 1/3$, the energy gap is proportional to $B_{\nu=\frac{1}{4}} - B$. This behavior has been confirmed in Ref. [27] as seen in Fig. 1.6.

Although some aspects of the theory of composite fermions have been verified in several experiments, further confirmation is still desirable since the experiments do not give us the ground state energy, ground state wave functions and only give limited

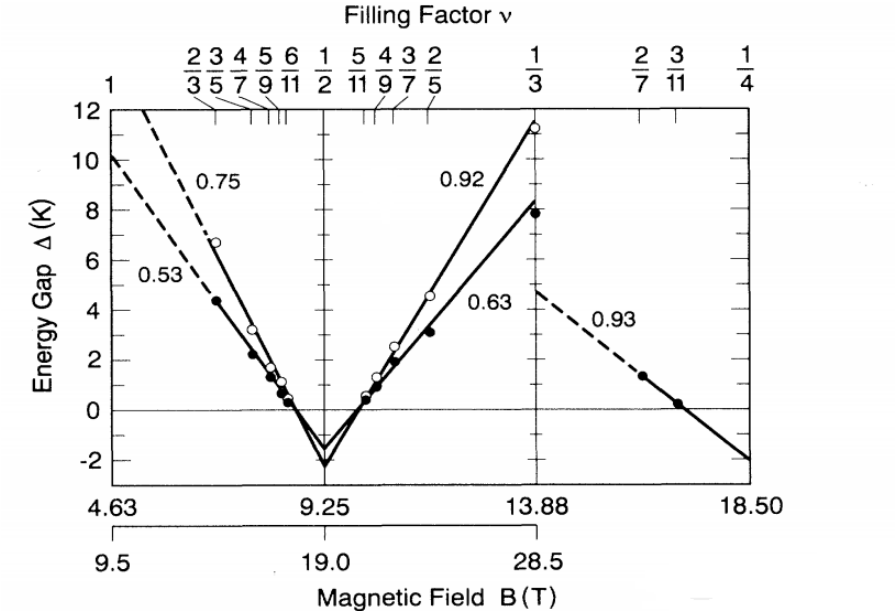

Figure 1.6: The linear fit of the gap versus the external magnetic field at various filling factors. Here gaps are determined from Eq. 1.2. The energy gap is found to be linear in $B - B_{\nu=\frac{1}{2}}$ when $1/3 < \nu < 1/2$ and linear in $B_{\nu=\frac{1}{2}} - B$ when $1/2 < \nu < 1$. The non-zero intercept at $\nu = 1/2$ is attributed to Λ level broadening. Here open and closed circles represent data from two different samples. The slopes of the linear lines can be used to determine the effective mass of composite fermions. Copyright © the American Physical Society. The right to use this figure from [27] has been granted by the American Physical Society.

information about the energy spectrum. In order to compare the energy spectrum of real quantum Hall systems and that of composite fermion theory, one must resort to exact diagonalization carried out by computers. This has been done by many groups[2] and it is found out that:

(1) The low energy spectrum of composite fermions is in good agreement with that of a real 2D electron system in a high magnetic field at various filling factors up to 10 particles.

(2) The overlaps of the trial wave function deduced from the theory of composite fermions and the ground state wave function of the fractional quantum Hall system are all between 0.994 and 1 up to 10 particles in the lowest Landau level.

Here the trial wave function used in the calculation of the overlap at $\nu = k/(2pk+1)$ is

$$\Psi = P_{LLL} \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^{2p} \Phi_k, \tag{1.48}$$

where Φ_k is the wave function of N electrons occupying the lowest k Landau levels and P_{LLL} means projection onto the lowest Landau level. The Jastrow factor $\prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^{2p}$ is used in Jain's trial wave function to account for the $2p\,\Phi_0$ flux quanta attached to each electron. Since Φ_k is already antisymmetric, the antisymmetry of Eq. 1.48 is guaranteed.

If $k = 1$, the wave function of N electrons occupying the lowest Landau level would be

$$\Phi_1 = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ \vdots & \vdots & & \vdots \\ z_1^{\frac{N}{2}-1} & z_2^{\frac{N}{2}-1} & \dots & z_N^{\frac{N}{2}-1} \end{vmatrix} \exp\left(-\frac{1}{4l_B^2} \sum_k |z_k|^2\right) \tag{1.49}$$

aside from a normalization factor. The determinant in the previous equation is exactly equal to $\prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)$. So we finally obtain the trial wave function of the fractional

[2]See references in Jain's monograph [37]

quantum Hall system at $\nu = 1/(2p+1)$:

$$\prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^{2p+1} \exp\left(-\frac{1}{4l_B^2} \sum_k |z_k|^2\right), \tag{1.50}$$

which is exactly the same as Laughlin's trial wave function in Eq. 1.5.

Similar to the fact that the Laughlin wave function is the zero mode(zero energy ground state) of the Haldane pseudopotential, the wave function before the projection to the lowest Landau level in Eq. 1.48 when $p = 1$ is also the zero mode of the two-body Trugman-Kivelson interaction

$$\nabla_1^2 \delta(x_1 - x_2)\, \delta(y_1 - y_2) \tag{1.51}$$

as easily seen in Appendix C. While when projected onto the lowest Landau levels, the Trugman-Kivelson interaction agrees with the V_1 Haldane pseudopotential, it is furthermore true that the unprojected Jain state at 2/5 is the zero mode of the Trugman-Kivelson interaction projected onto the lowest two Landau levels. This can be seen from Ref. [74].

1.11 The structure of this dissertation

The remainder of this dissertation is organized as follows: Chapter 2 reports work published in Ref. [22], which studies the zero modes of Haldane pseudopotentials using a second-quantized, algebraic approach without making contact with the first quantized versions of Haldane pseudopotentials and corresponding polynomial wave functions. In Chapter 3, the same algebraic approach will be used to study the Trugman-Kivelson interaction projected onto two Landau levels. This reveals new insights into the guiding-center structure of the unprojected Jain 2/5 state, in particular a system of root patterns following a generalized Pauli principle that are in full one-to-one correspondence with the edge mode counting. This formalism helps to

make precise in an algebraic way the notion of a composite fermion. Work in Chapter 3 is hitherto unpublished, and was carried out by the author of this dissertation in collaboration with Sumanta Bandyopadhyay and Professor Alexander Seidel. Finally, a summary of this dissertation will be given in Chapter 4.

Chapter 2

An Algebraic Approach to The Study of Zero Modes of Haldane Pseudopotentials

In Chapter 1, a brief review of integer and fractional quantum Hall effect has been given, including the experimental facts and their theoretical explanations. The emphasis there is on the first quantized model wave functions and their parent Hamiltonians. In this chapter, the lattice Hamiltonians arising from putting Haldane pseudopotentials into a second quantized or "guiding-center-only" form will be considered. These Hamiltonians are fascinating in the sense that they belong to a special class of Hamiltonians known as frustration free lattice Hamiltonians. We will show that the properties of "lattice versions" of pseudopotentials can be understood from a polynomial-free, intrinsically lattice point of view by constructing these pseudopotentials from an algebra of simple operators. We will also show that zero mode properties and hence the frustration free character of these lattice Hamiltonians derive from algebraic structures that these operators are part of. Our results will deepen insights into parent Hamiltonians of matrix product states with infinite bond dimensions, as could be of use, especially, in the study of fractional Chern insulators.

To begin with, we will present in Section 2.1 a motivation for our algebraic ap-

proach to the problem studied. We will then demonstrate the general properties of zero modes of frustration free lattice Hamiltonians in Section 2.2.1 and continue to construct recursively the zero modes of the lattice Hamiltonians pertaining to Haldane pseudopotential in Section 2.2.2. The construction relies largely on the algebra of operators in Section 2.2.3. The proof that zero modes thus constructed are indeed the zero modes of the Hamiltonian will be shown in Section 2.2.4. The uniqueness of the zero mode at the densest filling fraction $1/M$ will be proved in Section 2.2.6.

2.1 Introduction

Exactly solvable models of quantum mechanical systems serve to corroborate many of the most fundamental paradigms for the behavior of quantum matter. While more often than not, one is interested in the behavior of systems that are far from solvable, powerful effective theories often flow from a deep understanding of few isolated special Hamiltonians whose key properties are known exactly. Similarly, while no exact solutions can be given for the electronic wave functions of any atom save the simplest, hydrogen, a great wealth of atomic physics, quantum chemistry, and solid state physics is fundamentally built upon the latter. In the most fortunate cases, the solutions of such special Hamiltonians can be obtained in a variety of different approaches, each revealing intricate underlying mathematical structures that may be useful in many contexts beyond the scope of the original problem. Already in his original work[78], Schrödinger gave both the analytical wave function solution to the harmonic oscillator, rooted in the wealth of knowledge on differential equations and special functions inherited from the 19th century, as well as an algebraic construction. It was the latter that has deeply influenced the development of many-body physics and quantum field theory. A little later, Pauli introduced a more algebraic approach for the hydrogen atom, emphasizing the role of the symmetry that is associated with the conservation of the Lenz vector[63]. Since the very early days, it has been a characteristic of quantum theory that we may often choose between a language of analytic

wave functions satisfying differential wave equations and, more generally, an algebraic description, where the fundamental object is the C*-algebra of observables. This dichotomy is apparent already in the different pictures of quantum physics associated with the names of Schrödinger and Heisenberg.

A particular niche of quantum many-body physics is defined by the description of fractional quantum Hall states and their rich phenomenology. Special Hamiltonians in the sense described above have played an important role in this field since Haldane pointed out[31] that Laughlin states are the exact ground states of certain pseudopotentials, and shortly thereafter, the potential relevant to the $\nu = 1/3$ state was characterized as a Landau level projected ultra-short-ranged interaction by Trugman and Kivelson[98]. Unlike in other related fields, however, the main focus has been on the construction of first quantized wave functions as pioneered by Laughlin, that are written down quite independent of any Hamiltonian principles and are either required to satisfy certain analytic "clustering properties"[49, 59, 69, 11, 12, 107, 108], and/or are obtained from a given conformal edge theory[59]. In the most fortunate cases, including Laughlin[49], Moore-Read[59], Read-Rezayi[69], and the Gaffnian state[87], the aforementioned analytic properties also lend themselves to the construction of suitable parent Hamiltonians. E.g., in the aforementioned case of the $\nu = 1/3$ Laughlin state, it is the characteristic property of the wave function to vanish as the third power of inter-particle distance, whenever two particles are approaching one-another, that allows for the construction of a local parent Hamiltonian. Since their construction is based on analytic many-particle wave functions, these Hamiltonians are usually defined in a first quantized Language. For example, the Hamiltonian stabilizing the $\nu = 1/M$ Laughlin state is given by

$$H_{\frac{1}{M}} = \sum_{\substack{0 \leq m < M \\ (-1)^m = (-1)^M}} \sum_{i<j} P_{ij}^m \tag{2.1}$$

where i and j are particle indices, and P_{ij}^m projects the pair of particles with indices i and j onto states with relative angular momentum m. Generalizations of this construction exist, e.g., for the Moore-Read state[30], the Read-Rezayi series[69], and

for the Gaffnian[87]. In all these cases, one obtains positive Hamiltonians, whose zero energy ground states (zero modes) must satisfy certain analytic clustering conditions, and are given by the special wave functions defining "incompressible" ground states for the respective quantum Hall phase, as well as quasihole type excitations. The counting statistics of the latter are fundamentally related to conformal edge theories[68, 58, 3].

In this chapter, we discuss and further develop an alternative – algebraic – route to the construction of zero modes of the Hamiltonian Eq. (2.1), which does not make direct contact with the analytic clustering properties of first quantized wave functions. We believe that this Hamiltonian is of such fundamental interest that the exploration of its inner workings through a different framework will shine new light on the deep mathematical structures underlying fractional quantum Hall states, and may ultimately lend itself to the construction of new parent Hamiltonians. To make the problem concrete, we note that Landau level projection was left implicit in Eq. (2.1), but is routinely enforced. The effect of Landau level projection is to fix the degrees of freedom of the system associated with dynamical momenta, which determine the structure of a given Landau level. This leaves as the effective degrees of freedom the guiding center coordinates, and leads to the usual representation of the Hilbert space as a one-dimensional (1D) "lattice". The orbitals associated with this lattice are Landau level states with guiding center coordinates characterized by a single integer, angular momentum-like quantum number. It makes sense to write out the Hamiltonian (2.1) in the second quantized form, making explicit the dynamics in this guiding center orbital occupation number basis:

$$\begin{aligned} H_{\frac{1}{M}} &= \sum_{\substack{0\le m<M \\ (-1)^m=(-1)^M}} \sum_R Q_R^{m\dagger} Q_R^m \,, \\ \text{where} \qquad Q_R^m &= \sum_x g_{R,x}^m c_{R-x} c_{R+x} \,. \end{aligned} \tag{2.2}$$

Here, the sum over R is over both integer and half-odd integer values of the "center of mass" of a pair of particles destroyed by Q_R^m. The sum over x is over integer (half-odd-integer) when R is integer (half-odd-integer). The form factors $g_{R,x}^m$ depend on

the geometry, which shall be the disk, sphere, or cylinder geometry for the purpose of this chapter.

Traditionally, the form (2.2) for the Haldane pseudopotentials (and its generalizations for n-body interactions) has been given preference mostly for numerical work, as it makes Landau level projection and thus the dimensional reduction of the Hilbert space explicit. Today, however, there is additional motivation to be interested in Hamiltonians of the form given by Eq. (2.2). The idea of using the second quantized form of quantum Hall type Hamiltonians to generate frustration free lattice models for exotic electronic states in solids has been advocated by Lee and Leinaas[51], and also in Ref. [81]. Here, the orbital basis acted upon by the operators Q_R^m are Wannier states. Additional indices can be added to make such models describe systems in more than one dimension. However, as pointed to by Qi[65], a natural mapping exists between Wannier states of two-dimensional Chern band and Landau level orbitals in the cylinder geometry. Such Chern bands, in particular if they are flat[91, 89, 61, 85, 101, 71, 100, 14, 103, 102, 110, 53, 97, 111, 52, 21, 77] (though strictly, this requires non-local hopping terms[21]), together with appropriate interactions may harbor the sought-after fractional Chern insulator. Common to all these applications in solids is the fact that the first quantized versions of the respective Hamiltonians, e.g., Eq. (2.1) and the analytic forms of traditional quantum Hall ground states, are essentially meaningless; only second quantized forms, such as, Eq. (2.2) and a purely "guiding center" presentation of the wave function have natural meaning. For these reasons, there is much renewed interest in the "lattice" variant of quantum Hall-type Hamiltonians, especially[50] the manifestly translationally invariant type associated with the cylinder geometry.

Moreover, it has recently been argued by Haldane[33] that the essence of quantum Hall states such as Laughlin states lies in their guiding center description. Here, we want to adopt the (according to our reading) same point of view that analytic properties of polynomial wave functions, while fundamentally related to conformal edge theories[59], are not fundamentally essential to the topological order of the state.

Finally, quantum Hall parent Hamiltonians in the second quantized form (2.2)

give rise to frustration free 1D lattice models. Indeed, any zero energy eigenstate of Eq. (2.2) must be a *simultaneous* zero energy eigenstate of each of the positive operators $Q_R^{m\dagger}Q_R^m$. Equivalently, the state must be annihilated by each of the operators Q_R^m, i.e., it must satisfy the zero mode condition

$$Q_R^m \left|\psi_0\right\rangle = 0 \text{ for all } R, m \text{ included in Eq. (2.1).} \tag{2.3}$$

There has been much interest in general properties of such frustration free lattice models recently, in both 1D and in higher dimensions[1, 29, 43, 16, 25, 112, 20, 57, 79, 17, 23], especially in connection with matrix-product like ground states such models may have. In particular, for the cylinder and torus geometries, the operators Q_R^m are related by lattice translations. Hence for these geometries in particular, the model has much in common with other frustration free 1D lattice models arising in solid state, e.g. magnetic context. However, the models of the form (2.2) are arguably harder to study. For frustration free models, the problem is to find the common ground state of all the local terms entering the full Hamiltonian that do not commute with each other, while the individual ground state space of *one* such local term is typically easy to characterize. For Eq. (2.2), already finding the ground state subspace of one operator $Q_R^{m\dagger}Q_R^m$ is a highly non trivial task, owing to the exponentially decaying but non-local character of each such term. This problem was solved in Ref. [62], for a general class of models of this type, by making contact with the integrable structure of the hyperbolic Richardson-Gaudin model. Here, however, the focus will be entirely on finding zero modes of the full Hamiltonian (2.2).

For the reasons given above, we would like to have strategies to treat Hamiltonians that are *given* in the form (2.2), detached from the context of Landau levels and the analytic structure of their first quantized wave functions. To the best of our knowledge, such strategies are currently lacking, despite the recently appreciated matrix product structure of the Laughlin ground states of Eq. (2.1)[28, 115]. We will restrict ourselves to the case where the coefficients $g_{R,x}^m$ correspond to Haldane pseudopotentials in the disk, sphere, or cylinder geometry. Then by construction,

the Hamiltonian Eq. (2.2) is frustration free, i.e., there are states satisfying the zero mode condition Eq. (2.3). The question we wish to answer is how this fact can be understood in terms of algebraic properties of the operators Q_R^m, $Q_R^{m\dagger}$. This is indeed far from obvious. For the model at hand, we could of course "go back" by making connection with the language of first quantized analytic Landau level wave functions. However, this would certainly preclude an understanding of the zero mode property in a manner that is more intrinsic to the second quantized form in which the model is presented in Eq. (2.2). Moreover, this approach would also require a large amount of ingenuity if we did not already know how to re-cast the model in its original first-quantized form. Indeed, in going from analytic wave functions to the second quantized "lattice" description, information about the dynamical momenta that determine the structure of a Landau level has been dropped. As discussed initially, after Landau level projection one is working in a Hilbert space $\mathcal{H}_\omega$ that is "guiding center only". In contrast, the original analytic wave functions live in a larger Hilbert space $\mathcal{H}$, which is isomorphic to $\mathcal{H}_\pi \otimes \mathcal{H}_\omega$, where $\mathcal{H}_\pi$ is associated with the dynamical momenta of the system. (See also Ref. [70] for a recent discussion). It is only when the embedding

$$\mathcal{H}_\omega \hookrightarrow \mathcal{H}\,, \tag{2.4}$$

which in principle can be done in infinitely many ways, is defined in exactly the right manner that we recover the analytical properties of the Laughlin state that made the model tractable to begin with[33]. Here, we do not wish to "look back" at the larger Hilbert space $\mathcal{H}$, but instead take on the model as given in Eq. (2.2), and find a way to understand its frustration free character in a manner that is intrinsic to the algebraic properties of the operators Q_R^m. It is our hope that this approach will eventually pave the road to an even larger class of frustration free lattice models.

2.2 Algebraic treatment of zero modes

2.2.1 General properties

In this subsection only, we will consider the general class of Hamiltonians given by (2.2), with interaction parameters $g^m_{R,x}$ not necessarily identical to those obtained from Haldane pseudopotentials. For such general Hamiltonians, many useful properties of zero modes are known from Ref. [62], under the proviso that such zero modes exist, i.e., that the Hamiltonian is frustration free (up to some filling factor). To state these properties, let us first make the mathematical setup more precise. First, we consider the Landau level "lattice" space as half-infinite, as it is natural to the disk geometry or that of a half-infinite cylinder. That is, orbitals created by the operators $c^\dagger_r$ are labeled by a non-negative integer r, and we will write all equations with the convention

$$c_r = c^\dagger_r \equiv 0 \ \text{ for } r < 0 \tag{2.5}$$

in mind. We note that the zero modes we consider will generally occupy only a finite range of orbitals, and thus remain zero modes whenever a sufficiently large cutoff in orbital space is introduced, where only orbitals below this cutoff are retained. Thus, while we will not explicitly work with such a cutoff, all of the following is equally relevant to the spherical geometry, where the Hilbert space dimension is generally finite. To this end, we note that each state will be characterized by a particle number N, and a "maximum occupied orbital" r_{max}, where

$$r_{\mathsf{max}} = \max\{r | \langle\psi| c^\dagger_r c_r |\psi\rangle \neq 0\}\,, \tag{2.6}$$

and we always leave the ψ-dependence of N and r_{max} implicit. We then define the filling factor as

$$\nu = \frac{N-1}{r_{\mathsf{max}}}\,, \tag{2.7}$$

where the -1 in the numerator takes into account the topological shift for Laughlin

states. We now introduce $f = 0$ ($f = 1$) for bosons (fermions) and assume that $(-1)^f = (-1)^M$ in Eq. (2.2). The symbols $g^m_{R,x}$ are expected to have the symmetry $g^m_{R,-x} = (-1)^f g^m_{R,x}$. Note that the sum over m in Eq. (2.2) runs over $D = (M - f)/2$ terms. We then define $M(R)$ as the $D \times D$ matrix

$$M(R)_{ij=0\ldots D-1} = \begin{cases} g^{2j+f}_{R,i+f} \text{ for } 2R \text{ even} \\ \\ g^{2j+f}_{R,i+\frac{1}{2}} \text{ for } 2R \text{ odd}\,, \end{cases} \tag{2.8}$$

where we use the convention

$$g^m_{R,x} = 0 \text{ for } R < |x|, \tag{2.9}$$

setting to zero all coefficients that act on unphysical orbitals with negative index. Thus, for given R, the matrix $M(R)$ contains the parameters determining the interaction at the D closest distances. Then, under the general condition that for all $R = 0, \frac{1}{2}, 1 \ldots$, the matrix $M(R)$ has the maximum rank possible given the constraint (2.9), the results of Ref. [62] give the following:

Theorem 1.a The Hamiltonian (2.2) has no zero modes with filling factor $\nu > 1/M$.

Theorem 1.b If the Hamiltonian (2.2) has a zero mode at filling factor $\nu = 1/M$, it is unique.

Both of these theorems are direct consequences of the following. We will say that an occupation number eigenstate $|\{n\}\rangle = |n_0, n_1, \ldots\rangle$ satisfies the "M-Pauli principle", in the sense of Ref. [11], if there is no more than 1 particle in any M consecutive orbitals. We define "inward-squeezing" operations[11] of the form

$$c^\dagger_j c^\dagger_i c_{i-d} c_{j+d} \tag{2.10}$$

where $i \leq j$ and $d > 0$. Then, we say that an occupation number eigenstate $|\{n_i\}\rangle$

can be obtained from an occupation number eigenstate $|\{n_i'\}\rangle$ by inward-squeezing if $|\{n_i\}\rangle$ can be obtained from $|\{n_i'\}\rangle$ (up to some normalization and phase) by repeated application of operations of the form (2.10) (i.e., center-of-mass conserving inward pair hopping processes). We can expand any given state $|\psi\rangle$ in occupation number eigenstates:

$$|\psi\rangle = \sum_{\{n_i\}} C_{\{n_i\}} |\{n_i\}\rangle \,. \tag{2.11}$$

Then we have the following[62]:

Theorem 1 If $|\psi\rangle$ is a zero mode of the Hamiltonian (2.2), and the corresponding matrices $M(R)$ satisfy the maximum rank criterion defined above, then any basis state $|\{n_i\}\rangle$ appearing in (2.11) with $C_{\{n_i\}} \neq 0$ can be obtained from a $|\{n_i'\}\rangle$ (depending on $|\{n_i\}\rangle$ in general) through inward squeezing, where $C_{\{n_i'\}} \neq 0$ and $|\{n_i'\}\rangle$ satisfies the M-Pauli principle.

It is then easy to see that Theorems 1.a and 1.b follow, respectively, from the observations that there is no $|\{n_i\}\rangle$ satisfying the M-Pauli principle at filling factor $> 1/M$, and exactly one such at filling factor $1/M$. The latter is the well-known thin torus limit[93, 72, 8, 81, 82, 9, 10, 83, 5, 7, 80, 47, 84, 104] or root partition[11, 12, 13, 62] of the $1/M$-Laughlin state, $10\ldots010\ldots010\ldots01$, where 1s are separated by exactly $M-1$ zeros.

We may also remark the following trivial observation:

Proposition 2 If the Hamiltonian (2.2) has zero modes at some filling factor ν^* and with particle number $N > 1$, then there are also zero modes at filling factors $\nu < \nu^*$.

This simply follows from the zero mode condition (2.14), together with the observation that $[Q_R^m, c_r] = 0$. Hence we can always generate new zero modes from old ones by acting with destruction operators c_r. In general, however, we may not hope to generate all possible zero modes in this way. For the special case of the Laughlin-state parent Hamiltonians only, a more complete prescription using second quantization was given in Ref. [62], where it was noted that certain particle number conserving operators generate new zero modes at higher r_{max} when acting on a given

zero mode $|\psi\rangle$.[1] A variant of these operators will be defined below.

It is worth emphasizing again that while all facts stated in this section are known for the Laughlin-state parent Hamiltonians (2.1) from first quantized wave function considerations[73] and known squeezing properties of "special" wave functions such as Laughlin states[72], all of the above was shown in Ref. [62] for zero modes of the more general class of Hamiltonians (2.2), under the general maximum rank condition stated above. This maximum rank condition is easily adapted to other generalized Pauli principles and n-body operators. None of this makes use of analytic clustering properties (which are in fact not necessary, as, e.g., demonstrated by the examples given in Refs. [60]). Given the above, it seems that most known facts about Laughlin-state parent Hamiltonians are already within reach of a purely algebraic, or second-quantized derivation. There is, however, a key ingredient thus far missing: Namely, the fact that there exists, to begin with, a zero mode at the special "incompressible" filling factor $1/M$. Once this is established, further zero modes can be generated using the operators defined in Ref. [62], or the operators given in Eq. (2.19) below. To understand the existence of a special zero mode at filling factor $1/M$ (whose uniqueness is then guarantied, e.g., by Theorem 1.b) in terms of algebraic properties of the operators Q_R^m is the main goal of this chapter.

2.2.2 Recursive definition of the Laughlin state in second quantization

The Laughlin state at filling factor $\nu = 1/M$ and other zero modes, which for $\nu < 1/M$ physically represent quasihole and edge excitations, can be characterized as forming the common null space of the operators Q_R^m for all $R = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ and all $m = M-2, M-4 \ldots 0$ (1) for bosons (fermions).

[1] This corresponds to the familiar multiplication with symmetric polynomials in first quantized language.

As explained in Ref. [62], the form factors $g^m_{R,x}$ may be taken to be of the form

$$g^m_x = x^m. \tag{2.12}$$

Formally, this corresponds to working with zero modes in the cylinder geometry in the limit where the cylinder radius goes to infinity. The zero modes in the disk geometry, that of any cylinder of finite thickness, or the sphere are in one-to-one correspondence with the zero modes obtained in this way[2], where for the sphere, an upper cutoff in r_{max} must be introduced. We also work with a lower cutoff in orbital space, corresponding to a half-infinite geometry, imposed by the condition (2.5). This condition will always be left understood in the following. We note that for a general cylinder with radius $R_y \equiv 1/\kappa$, the coefficient g^m_x corresponding to the mth Haldane pseudopotential is given by[62]:

$$g^m_x = \mathcal{N}_m \sqrt{\kappa}\, H_m(\sqrt{2}\kappa x) e^{-\kappa^2 x^2}, \tag{2.13}$$

where $\mathcal{N}_m$ is a normalization factor, and H_m is the mth Hermite polynomial. The simple limiting form (2.12) is justified by first taking linear combinations in m of the Q^m_R at *finite* κ such that the polynomials in (2.13) are replaced by monomials, and then taking the limit $\kappa \to 0$. We note that forming such linearly independent new linear combinations does not affect the common null space of the operators Q^m_R. As explained in the preceding paragraph, we now focus on the question of the existence of a zero mode at filling factor $\nu = 1/M$ satisfying

$$\begin{aligned} Q^m_R |\psi_N\rangle = 0 \quad &\text{for all } R, \text{ and for } 0 \le m < M \\ &m \equiv M \mod 2. \end{aligned} \tag{2.14}$$

Here, the subscript N stands for an N-particle state with $r_{\mathsf{max}} = M(N-1)$, whose existence we will prove inductively. We also introduce the "angular momentum"

[2]See. Ref. [62] for a second-quantized proof.

operator

$$L = \sum_r r c_r^\dagger c_r, \tag{2.15}$$

and anticipate that $|\psi_N\rangle$ will be an L-eigenstate with eigenvalue $L = \frac{1}{2}MN(N-1)$, as befits a $\nu = 1/M$ Laughlin state.

We seek a recursive definition for $|\psi_N\rangle$ for which we can prove the zero mode property inductively. Our general strategy will be the following. We start with the trivial identity

$$|\psi_N\rangle = \frac{1}{N}\sum_r c_r^\dagger c_r |\psi_N\rangle\,. \tag{2.16}$$

We have observed above already that if $|\psi_N\rangle$ is an N-particle zero mode, then $c_r|\psi_N\rangle$ is a zero mode with $N-1$ particles. As such, it can be generated from the $N-1$ particle incompressible Laughlin state at $\nu = 1/M$ through the application of an appropriate particle number *conserving* operator that creates zero energy edge excitations. We thus conjecture that there is a well-defined operator P_ℓ, creating an edge excitation that increases the angular moment L by ℓ units while conserving particle number, such that

$$c_r|\psi_N\rangle = P_{M(N-1)-r}|\psi_{N-1}\rangle \tag{2.17}$$

holds. Here, we have used the fact that the incompressible Laughlin state at filling factor $1/M$ has L-eigenvalue $\frac{1}{2}MN(N-1)$. This leads to a recursive definition of the N-particle Laughlin state in terms of the $N-1$-particle Laughlin state:

$$|\psi_N\rangle = \frac{1}{N}\sum_{r\geq 0} c_r^\dagger P_{M(N-1)-r}|\psi_{N-1}\rangle\,. \tag{2.18}$$

The strategy will hence be to identify the precise form of the operator P_ℓ, and prove inductively that indeed Eq. (2.18) defines an N-particle zero mode at filling factor $1/M$, as long as $|\psi_{N-1}\rangle$ has the same property for $N-1$ particles. We note that the strategy given here is quite general, and the same logic would in principle lead to a recursive expression similar to Eq. (2.18) for other types of quantum Hall states. We focus on the case of the Laughlin state here. In this special case, the relation (2.18) we seek turns out to be a second quantized rendering of Read's recursive formula

for the Laughlin state using the "string order parameter"[67]. The relation of our formalism to the string order parameter is in [55] and will not be elaborated further in the remaining of this chapter.

2.2.3 The generators of edge excitations and relevant commutation relations

As motivated above, we are interested in particle number conserving operators that generate new zero modes when acting on given zero modes. We will think of such operators as generators of (zero energy) edge excitations. Consider the operators

$$e_n = \frac{1}{n!} \sum_{i_1 \geq 0, i_2 \geq 0, \ldots i_n \geq 0} c^\dagger_{i_1+1} c^\dagger_{i_2+1} \ldots c^\dagger_{i_n+1} c_{i_n} \ldots c_{i_2} c_{i_1} \,, \tag{2.19}$$

where the operators c_r, $c_r^\dagger$ satisfy standard bosonic (fermionic) commutation (anti-commutation) relations for M even (odd), and we fix an integer $M > 1$ here and in the following. We also define $e_0 = \mathbb{1}$ and $e_n = 0$ for $n < 0$. These operators conserve particle number, and have the property that if $|\psi\rangle$ is a zero mode, then so is $e_n|\psi\rangle$. This follows since by definition, a zero mode is annihilated by all the operators Q^m_R, with m and R as in Eq. (2.14), and we have the commutator

$$\begin{aligned}[Q^m_R, e_n] =& e_{n-2} Q^m_{R-1} + \\ & e_{n-1} \sum_{\substack{0 \leq k \leq m-f \\ (-1)^k = 1}} \binom{m}{k} 2^{1-k} Q^{m-k}_{R-1/2} \,,\end{aligned} \tag{2.20}$$

which vanishes upon acting on all zero modes. The operators e_n are not the same edge mode generators as those defined in Ref. [62]. The relation between the latter and the e_n is not of any importance in the following, and has been clarified in [55]. In terms of the e_n, we now define new operators

$$P_\ell = (-1)^\ell \sum_{n_1 + n_2 + \cdots + n_M = \ell} e_{n_1} e_{n_2} \ldots e_{n_M} . \tag{2.21}$$

The latter are likewise particle number conserving generators of zero modes, since the e_n have this property. These operators P_ℓ depend on M as seen from the definition, but we leave the dependence implicit. It is also understood that $P_0 = \mathbb{1}$ and $P_l = 0$ for $l < 0$. It is furthermore easy to see that P_ℓ raises the angular momentum by ℓ. We now *define* the $\nu = 1/M$ Laughlin state through Eq. (2.18), where

$$|\psi_{N=0}\rangle = |0\rangle\,, \tag{2.22}$$

$|0\rangle$ being the vacuum, which also leads to $|\psi_{N=1}\rangle = c_0^\dagger|0\rangle$. For the time being, we *assert* that Eq. (2.17) follows from this definition and from Eq. (2.21). It turns out that most technical difficulties can be attributed to the proof of this assertion, which we relegate to Sec.2.2.5. Our key result, namely, that Eq. (2.18) defines a zero mode at filling factor $1/M$, follows rather easily from Eq. (2.17) and the following commutation relations:

$$[Q_R^m, c_r] = 0, \tag{2.23}$$

$$[Q_R^m, c_r^\dagger] = 2(-1)^f (R-r)^m c_{2R-r}, \tag{2.24}$$

$$[c_r, P_l] = \sum_{1\le k\le M} (-1)^k \binom{M}{k} P_{l-k} c_{r-k}, \tag{2.25}$$

$$[P_l, c_r^\dagger] = \sum_{1\le k\le M} (-1)^k \binom{M}{k} c_{r+k}^\dagger P_{l-k}, \tag{2.26}$$

and

$$[e_m, e_n] = 0. \tag{2.27}$$

Another useful way to write Eq. (2.25) is

$$c_r P_l = \sum_{0\le k\le M} (-1)^k \binom{M}{k} P_{l-k} c_{r-k}, \tag{2.28}$$

and similarly for Eq. (2.26).

2.2.4 Proof of zero mode property of $|\psi_N\rangle$

We proceed by showing that given all of the above, and assuming Eq. (2.17) to be true for now, it follows that Eqs. (2.18), (2.22) define a non-vanishing zero mode at the "incompressible" filling factor $1/M$.

To see that Eq. (2.18) gives a zero mode, we proceed inductively, starting with $N = 2$:

$$\begin{aligned}
&|\psi_{N=2}\rangle \\
&= \frac{1}{2}\sum_{r\ge 0} c_r^\dagger P_{M-r} c_0^\dagger |0\rangle \\
&= \frac{1}{2}\sum_{r\ge 0} c_r^\dagger \sum_{0\le k\le M} (-1)^k \binom{M}{k} c_k^\dagger P_{M-r-k}|0\rangle\,,
\end{aligned}$$

where the last line follows from Eq. (2.26). For positive index, P_{M-r-k} is the sum of products of e operators that have annihilation operators on the right, thus $P_{M-r-k}|0\rangle$ gives zero unless $M - r - k = 0$. Therefore

$$|\psi_{N=2}\rangle = \frac{1}{2}\sum_{r\ge 0}(-1)^{M-r}\binom{M}{r} c_r^\dagger c_{M-r}^\dagger |0\rangle. \tag{2.29}$$

It is easy to show that

$$\begin{aligned} & Q_R^m|\psi_{N=2}\rangle \\ & = \delta_{M,2R}(-1)^{3R} \sum_{-R\le x\le R} x^m(-1)^x \binom{2R}{R+x}|0\rangle \\ & = \delta_{M,2R}(-1)^M \sum_{0\le x'\le M} \left(x' - \frac{M}{2}\right)^m (-1)^{x'} \binom{M}{x'}|0\rangle \\ & = 0\,, \end{aligned} \tag{2.30}$$

where the sum in the last line follows from the fact that [3]

$$\sum_{0\le j\le M} j^m(-1)^j \binom{M}{j} = 0 \quad \text{for } 0 \le m < M\,. \tag{2.31}$$

Thus

$$Q_R^m|\psi_{N=2}\rangle = 0. \tag{2.32}$$

Now assume

$$\begin{aligned} Q_R^m|\psi_{N-1}\rangle = 0 \quad & \text{for all } R, \text{ and for } 0 \le m < M \\ & m \equiv M \mod 2\,. \end{aligned} \tag{2.33}$$

[3] Let $[j]_0 = 1$, $[j]_m = j(j-1)\dots(j-m+1)$ for $m > 0$. Then $\sum_{0\le j\le M}[j]_m(-1)^j\binom{M}{j} = \sum_{m\le j\le M}\frac{j!}{(j-m)!}(-1)^j\binom{M}{j} = (-1)^m[M]_m(1-1)^{M-m}$. This gives 0 for $0 \le m < M$. Since $[j]_m$ is clearly an mth degree polynomial in j, we can make new linear combinations of the latter identities to obtain Eq. (2.31). Extending consideration to $m = M$ gives all the ingredients for the interesting identity [75] $\sum_{j=0}^M(-1)^j\binom{M}{j}(x-j)^M = M!\quad \forall x$.

We have

$$
\begin{aligned}
&Q_R^m|\psi_N\rangle \\
&= \frac{1}{N}\sum_{r\geq 0}(c_r^\dagger Q_R^m + 2(-1)^f(R-r)^m c_{2R-r})P_{M(N-1)-r}|\psi_{N-1}\rangle \\
&= \frac{1}{N}\sum_{r\geq 0}2(-1)^f(R-r)^m c_{2R-r}P_{M(N-1)-r}|\psi_{N-1}\rangle \\
&= \frac{1}{N}\sum_{r\geq 0}2(-1)^f(R-r)^m c_{2R-r}c_r|\psi_N\rangle \\
&= \frac{2}{N}Q_R^m|\psi_N\rangle\,,
\end{aligned}
$$

where we have used Eq. (2.18) and Eq. (2.24) to get the second line. The third line uses the fact that $P_{M(N-1)-r}|\psi_{N-1}\rangle$ also satisfies the zero mode condition (2.33), since P_ℓ, being a product of e-operators, generates new zero modes from old ones. The fourth line follows from Eq. (2.17). Therefore, for $N \geq 3$, if $|\psi_{N-1}\rangle$ satisfies the zero mode condition, so will $|\psi_N\rangle$. Finally, $|\psi_{N=0}\rangle$ and $|\psi_{N=1}\rangle$ are trivially zero modes. Thus, all $|\psi_N\rangle$ satisfy the zero mode property (2.14). We will still need to demonstrate that $|\psi_N\rangle$ has filling factor $1/M$, and in particular does not vanish for any N. Before doing so in Sec.2.2.6, we attend to the technical heart of the proof, Eq. (2.17).

2.2.5 Expressing electron holes through edge excitations

We note that Eq. (2.17) expresses an electron hole inserted into an N-particle Laughlin state through a superpositions of general edge excitations created on top of an $N-1$ particle Laughlin state. We believe that this relation could prove useful in itself beyond the application given here. We prove Eq. (2.17) inductively: For $N = 1$,

$$c_r|\psi_{N=1}\rangle = P_{-r}|0\rangle \tag{2.34}$$

is satisfied for $r = 0$, and for $r \neq 0$ both sides vanish identically. Now for some $N > 1$, we make the assumption that

$$c_r|\psi_{N-1}\rangle = P_{M(N-2)-r}|\psi_{N-2}\rangle, \tag{2.35}$$

The definition of $|\psi_N\rangle$, Eq. (2.18), gives

$$\begin{aligned} c_r|\psi_N\rangle &= c_r \frac{1}{N} \sum_{r' \geq 0} c_{r'}^\dagger P_{M(N-1)-r'}|\psi_{N-1}\rangle \\ &= \frac{1}{N} \sum_{r' \geq 0} (\delta_{rr'} + (-1)^f c_{r'}^\dagger c_r) P_{M(N-1)-r'}|\psi_{N-1}\rangle \,. \end{aligned} \tag{2.36}$$

Employing Eq. (2.28), the last term of Eq. (2.36)

$$\frac{1}{N} \sum_{r'} (-1)^f c_{r'}^\dagger c_r P_{M(N-1)-r'}|\psi_{N-1}\rangle \tag{2.37}$$

is found to be

$$\begin{aligned} &\frac{(-1)^f}{N} \sum_{0 \leq k \leq M-1} (-1)^k \binom{M}{k} \sum_{r'} c_{r'}^\dagger P_{M(N-1)-r'-k} c_{r-k}|\psi_{N-1}\rangle \\ &+ \frac{1}{N} \sum_{r'} c_{r'}^\dagger P_{M(N-1)-r'-M} c_{r-M}|\psi_{N-1}\rangle, \end{aligned} \tag{2.38}$$

where we split off the last term. We now use the induction assumption, according to which $c_{r-M}|\psi_{N-1}\rangle$ is equal to $P_{M(N-1)-r}|\psi_{N-2}\rangle$. Therefore the last term of Eq. (2.38) can be further simplified to read

$$\frac{1}{N} \sum_{r'} c_{r'}^\dagger P_{M(N-2)-r'} P_{M(N-1)-r}|\psi_{N-2}\rangle. \tag{2.39}$$

Here, we may now change the order of $P_{M(N-2)-r'}$ and $P_{M(N-1)-r}$, since they are

products of commuting e_n's. Then we can use Eq. (2.26) to rewrite

$$\frac{1}{N}\sum_{r'} c^{\dagger}_{r'} P_{M(N-1)-r} P_{M(N-2)-r'} |\psi_{N-2}\rangle \tag{2.40}$$

as

$$\begin{aligned}\frac{1}{N} P_{M(N-1)-r} \sum_{r'} c^{\dagger}_{r'} P_{M(N-2)-r'} |\psi_{N-2}\rangle - \frac{1}{N} \sum_{r'} \\ \times \sum_{1\le i\le M} (-1)^i \binom{M}{i} c^{\dagger}_{r'+i} P_{M(N-1)-r-i} P_{M(N-2)-r'} |\psi_{N-2}\rangle,\end{aligned} \tag{2.41}$$

where the first term is just

$$\frac{N-1}{N} P_{M(N-1)-r} |\psi_{N-1}\rangle \tag{2.42}$$

utilizing our induction assumption and the second term can be written as

$$\begin{aligned}\frac{1}{N} \sum_{0\le k\le M-1} (-1)^{k+M+1} \binom{M}{k} \\ \times \sum_{r''} c^{\dagger}_{r''} P_{M(N-2)-r+k} P_{M(N-1)-r''-k} |\psi_{N-2}\rangle\end{aligned} \tag{2.43}$$

after we make changes of variables $r'' = r' + i$ and $k = M - i$. Eq. (2.43) is seen to cancel the first term of Eq. (2.38) after we change the order of two P operators and use once more our induction assumption. Finally we get

$$\begin{aligned} c_r |\psi_N\rangle =& \left(\frac{1}{N} + \frac{N-1}{N}\right) P_{M(N-1)-r} |\psi_{N-1}\rangle \\ =& P_{M(N-1)-r} |\psi_{N-1}\rangle, \end{aligned} \tag{2.44}$$

thus completing our induction to prove Eq. (2.17).

2.2.6 Properties of $|\psi_N\rangle$

In the above, we have shown that the recursively defined state (2.18) has the zero mode property for all N. The proof was based solely on the algebraic properties described in Sec.2.2.3. To achieve our initial goal, we must also demonstrate that $|\psi_N\rangle$ is a *non-vanishing* N-particle state at filling factor $1/M$.

To this end, we define the "thin cylinder state" $|\tilde{\psi}_N\rangle$ discussed below Theorem 1 via

$$|\tilde{\psi}_N\rangle = |10\ldots010\ldots010\ldots\rangle, \tag{2.45}$$

where exactly N 1's are separated by sequences of $M-1$ zeros. We then assert the following

Proposition 3 The state $|\psi_N\rangle$ defined by Eq. (2.18) is dominated by the basis state $|\tilde{\psi}_N\rangle$ with $\langle\tilde{\psi}_N|\psi_N\rangle \neq 0$. Here, the notion of dominance means, as usual [11, 12], that all basis states appearing in the expansion (2.11) can be obtained from $|\tilde{\psi}_N\rangle$ via inward squeezing operations, as explained following Eq. (2.10). The proposition in particular implies all the desired information about $|\psi_N\rangle$. It clearly implies that $|\psi_N\rangle$ is non-zero. It is also easy to see, given the definition (2.7), that any state dominated by $|\tilde{\psi}_N\rangle$ has a filling factor of at most $1/M$, and has precisely filling factor $1/M$ if $\langle\tilde{\psi}_N|\psi_N\rangle \neq 0$.

Again, we prove Proposition 3 inductively. For $N = 1$, the statement is obvious. Assuming that Proposition 3 has been proven for $N-1$ with $N \geq 2$, we consider $|\psi_N\rangle$ as defined through Eq. (2.18). It is elementary to see from this equation that if $|\psi_{N-1}\rangle$ is dominated by the basis state $|\tilde{\psi}_{N-1}\rangle$, then $|\psi_N\rangle$ must have *at least* filling factor $1/M$, i.e., its r_{max} can be at most $M(N-1)$. On the other hand, by Theorem 1.a, since we know that $|\psi_N\rangle$ has the zero mode property, it must have a filling factor of *exactly* $1/M$, so long as it is non-zero. If indeed $|\psi_N\rangle$ is non-zero, by Theorem 1 its expansion (2.11) into occupation number eigenstates must then also be dominated by $|\tilde{\psi}_N\rangle$, the latter being the only such state that satisfies the M-Pauli principle at filling factor $1/M$. Therefore, all that remains to show is that $\langle\tilde{\psi}_N|\psi_N\rangle \neq 0$.

By the induction assumption, we may write

$$|\psi_{N-1}\rangle = C_{\tilde{\psi}_{N-1}}|\tilde{\psi}_{N-1}\rangle + \sum_{|\{n_i\}\rangle \neq |\tilde{\psi}_{N-1}\rangle} C_n|\{n_i\}\rangle \tag{2.46}$$

with $C_{\tilde{\psi}_{N-1}} \neq 0$, and every $|n\rangle$ appearing in the sum being dominated by $|\tilde{\psi}_{N-1}\rangle$. When this is plugged into Eq. (2.18), one may see that

$$\begin{aligned}
&\frac{1}{N}\sum_{r\geq 0} c_r^\dagger P_{M(N-1)-r}|\tilde{\psi}_{N-1}\rangle \\
=&\frac{1}{N}\sum_{r\geq 0} c_r^\dagger P_{M(N-1)-r} c_0^\dagger c_M^\dagger \cdots c_{(N-2)M}^\dagger|0\rangle \\
=&\frac{1}{N}\sum_{0\leq k_0,k_1,\ldots,k_{N-2}\leq M} c^\dagger_{(N-1)M-k_0-\ldots-k_{N-2}} c_{k_0}^\dagger c_{k_1+M}^\dagger \cdots \\
&\times c^\dagger_{k_{N-2}+(N-2)M}|0\rangle
\end{aligned} \tag{2.47}$$

after using the same approach leading to Eq. (2.29). It is clear now why r_{max} can be at most $M(N-1)$. To generate $|\tilde{\psi}_N\rangle$, $(N-1)M - k_0 - \cdots - k_{N-2}$ should be equal to mM where $0 \leq m \leq N-1$ and $k_0, k_1+M, \ldots, k_{N-2}+(N-2)M$ each should assume one of the values $0, M, \ldots, (m-1)M, (m+1)M, \ldots, (N-1)M$.

If $m < N-1$, the only index that can assume the value $(N-1)M$ is $k_{N-2}+(N-2)M$, and this fixed $k_{N-2} = M$. Working our way down in j from $j = N-2$, we find from the same reasoning that $k_j = M$ for $j \geq m$. Then, in order for the first index $(N-1)M - k_0 - \cdots - k_{N-2}$ to equal mM, all the remaining k_j for $0 \leq j < m$ must vanish.

The only solution for given m is thus $k_0 = k_1 = \cdots = k_{m-1} = 0$ and $k_m = k_{m+1} = \cdots = k_{N-2} = M$. Therefore $|\tilde{\psi}_N\rangle$ is generated from $|\tilde{\psi}_{N-1}\rangle$ through N possible choices of m all leading to the same coefficient of $|\tilde{\psi}_N\rangle$, which is $(-1)^{(N-1)M}$. Furthermore, the states dominated by $|\tilde{\psi}_{N-1}\rangle$ cannot generate $|\tilde{\psi}_N\rangle$. To see this, we

act $\frac{1}{N}\sum_{r\geq 0} c_r^\dagger P_{M(N-1)-r}$ on one of those states:

$$\begin{aligned}
&\frac{1}{N}\sum_{r\geq 0} c_r^\dagger P_{M(N-1)-r}|n\rangle \\
=&\frac{1}{N}\sum_{r\geq 0} c_r^\dagger P_{M(N-1)-r} c_{r_0}^\dagger c_{r_1}^\dagger \dots c_{r_{N-2}}^\dagger|0\rangle \\
=&\frac{1}{N}\sum_{0\leq k_0,k_1,\dots,k_{N-2}\leq M} c^\dagger_{(N-1)M-k_0-\dots-k_{N-2}} c^\dagger_{k_0+r_0} c^\dagger_{k_1+r_1}\cdots \\
&\times c^\dagger_{k_{N-2}+r_{N-2}}|0\rangle\,.
\end{aligned} \tag{2.48}$$

We may assume the r_j to be in ascending order. $|\{n_i\}\rangle$ is obtained from $|\tilde{\psi}_{N-1}\rangle$ through inward squeezing defined in Eq. (2.10). The largest index j_0 for which r_j differs from Mj must then have $r_{j_0} < Mj_0$. We may now ask if Eq. (2.48) could make contributions to $|\tilde{\psi}_N\rangle$. We can follow the same logic as above, fixing the first index to be mM, and then fixing the k_j, starting with $j = N-2$ and working our way down. We will always find a contradiction once we reach j_0, using $r_{j_0} < Mj_0$.

In all, we find $C_{\tilde{\psi}_N} = (-1)^{(N-1)M} C_{\tilde{\psi}_{N-1}}$, which does not vanish. This completes the argument that $\langle\tilde{\psi}_N|\psi_N\rangle \neq 0$.

2.3 Discussion and Conclusion

In this chapter, we have studied a particular class of frustration free lattice Hamiltonians arising in the fractional quantum Hall effect via Haldane's pseudopotential formalism. There is a great wealth of knowledge on general properties of frustration free Hamiltonians, which recently was further field by interest in matrix product states. In general, the interrelation between matrix product states on a lattice and their frustration free parent Hamiltonians is well understood[29]. This, however, is arguably different for the class we have studied here, which differs from most examples in the literature in that the interaction is not strictly local. Despite the fact that the matrix product structure of the Laughlin state has recently been

appreciated[28, 115], to the best of our knowledge, the only way to understand the frustration free character of its "lattice" parent Hamiltonians involved going back to first quantization – using a language of analytic wave functions where the "lattice character" is lost, as we reviewed initially. While the first quantized language of polynomial wave functions is powerful and has thus far been a primary driving factor in this field, it is a priori not clear why the inclusion of additional degrees of freedom (dynamical momenta) is necessary to solve the problem of studying the zero modes of the lattice Hamiltonians. Such a point of view is natural and may be of great value especially in the study of parent Hamiltonians for fractional Chern insulators[91, 89, 61, 85, 101, 71, 100, 14, 103, 102, 110, 53, 97, 111, 52, 21, 77]. In this work, we have demonstrated that the frustration free character of "lattice", or second quantized, Haldane-type pseudopotential Hamiltonians can be understood directly, without making contact with a polynomial language. To this end, we explicitly constructed the $1/M$ Laughlin state in the lattice basis. This was done by iteratively constructing the N-particle Laughlin state from the $N-1$ particle one, in what turned out to be a second quantized form of Read's iterative formula using the order parameter for Laughlin states[67]. We have identified the proper algebra of lattice operators that allows both construction of the Laughlin state and statement of the zero mode condition. Using this algebra alone we have demonstrated that the "lattice" Hamiltonians obtained from first quantized Laughlin state parent Hamiltonians have a unique zero mode at the respective highest filling factor $1/M$. From this very fact the entire zero mode structure can be derived, also by using only the second quantized algebraic setting used here[62, 55].

Chapter 3

The Study of Second Quantized Trugman-Kivelson Interaction as A Pseudopotential in Fractional Quantum Hall Regime

In Section 1.10, we have introduced the concept of composite fermions and demonstrated its power in explaining filling factors of the form $k/(2pk \pm 1)$ observed in experiments. A series of unprojected Jain state at $k/(2k+1)$ is of special interest since they are zero modes of the Trugman-Kivelson interaction projected to lowest k Landau levels. The Trugman-Kivelson is equivalent to the V_1 Haldane pseudopotential when projected to the lowest Landau levels[31] and hence has the Laughlin 1/3 state as the highest density zero mode in the lowest Landau level. However, we do not know for sure whether the unprojected Jain 2/5 state, as the zero mode of the Trugman-Kivelson interaction when projected to the lowest two Landau levels, is the densest and the unique zero mode.

In this chapter, we will study the two-body Trugman-Kivelson interaction[98]

$$H = P_n \nabla_1^2 \delta(x_1 - x_2)\, \delta(y_1 - y_2)\, P_n\,, \tag{3.1}$$

projected onto the first n Landau levels via an orthogonal projection operator P_n focusing on the case where $n = 2$. For $n = 1$, it is well known that this interaction agrees, up to a factor, with the V_1 Haldane pseudo potential[31]. The case $n = 2$ was identified by Rezayi and MacDonald [74] to give rise to a parent Hamiltonian for the Jain 2/5 state, where at the same time, the kinetic energy is quenched not only within individual Landau levels, but the splitting between the lowest and the first excited Landau level is being ignored[74]. Here we wish to establish and discuss the complete zero mode properties of this Hamiltonian.

3.1 Second quantization on disk

We begin by establishing a second quantized form of the Hamiltonian in various geometries, starting with the disk geometry. As demonstrated in Chapter 2, for positive, angular momentum conserving two-particle operators, the second quantized many-body Hamiltonian is generally of the form

$$H = \sum_{k=1}^{M} \sum_{R} T_R^{(k)\dagger} T_R^{(k)} , \tag{3.2}$$

where

$$T_R^{(k)} = \sum_{x} f_{i,j}^{k}(R, x) c_{i,R-x} c_{j,R+x} \tag{3.3}$$

destroys a pair of particles with well defined angular momentum $2R$, $c_{i,m}$ is an electron destruction operator for a state in the i-th Landau level (LL) with angular momentum m, and $f_{i,j}^{k}(R, x)$ is a form factor defining the operator $Q_R^{(k)}$. In Eq. (3.2), The sum over R is over integer and half-odd integer values, and x in the definition of $A_R^{(k)}$ is either over integer or over half-odd integer, depending on R (i.e., $2x \equiv 2R \mod 2$) so that orbital indices $R \pm x$ are integers.

We now work out the connection between Eqs. (3.1) and (3.2) and specialize to $n = 2$ Landau levels (carrying Landau level indices 0 and 1, respectively, in the following). To this end, we recall that in Appendix A, we have shown the wave

functions for a single particle in the disk with angular momentum $L_z = m$ in the lowest and the first excited Landau levels under symmetric gauge,

$$\eta_{0,m}(z) = \frac{z^m e^{-|z|^2/4l_B^2}}{\sqrt{2\pi 2^m l_B^{2m+2} m!}} \tag{3.4}$$

and

$$\eta_{1,m}(z) = \frac{(\bar{z} z^{m+1} - 2l_B^2(m+1)z^m)\, e^{-|z|^2/4l_B^2}}{\sqrt{2\pi 2^{m+2} l_B^{2m+6}(m+1)!}} \tag{3.5}$$

respectively, where $z = x + i\ y$ is the complex coordinate on the disk, and l_B is magnetic length $\sqrt{\hbar/eB}$ which is set equal to unity for simplicity. As an immediate consequence, we have the following analytic structure for general *two-particle* wave functions projected onto the first two Landau levels,

$$\begin{aligned}\psi(z_1, z_2) =& \Big(C_{00}(z_1, z_2) + \bar{z}_1 C_{10}(z_1, z_2) + \bar{z}_2 C_{01}(z_1, z_2) \\ &+ \bar{z}_1 \bar{z}_2 C_{11}(z_1, z_2) \Big) \\ &\times e^{-\frac{|z_1|^2}{4l_B^2} - \frac{|z_2|^2}{4l_B^2}},\end{aligned} \tag{3.6}$$

where $C_{00}(z_1, z_2), C_{10}(z_1, z_2), C_{01}(z_1, z_2)$ and $C_{11}(z_1, z_2)$ are holomorphic functions of z_1 and z_2. For two-particle states, it is further generally advantageous to phrase expressions in terms of a center-of-mass coordinate $z_c = (z_1 + z_2)/2$ and a relative coordinate $z_r = z_1 - z_2$, and their complex conjugates $\bar{z}_c$ and $\bar{z}_r$. Furthermore, in this chapter we will be exclusively considering fermions. Then, Eq. (3.6) can be recast as

$$\begin{aligned}\psi(z_c, z_r) =& \Big(d_{00}(z_c, z_r) + \bar{z}_c d_{10}(z_c, z_r) + \bar{z}_r d_{01}(z_c, z_r) \\ &+ (\bar{z}_c^2 - \bar{z}_r^2/4) d_{11}(z_c, z_r) \Big) \\ &\times e^{-\frac{|z_c|^2}{2l_B^2} - \frac{|z_r|^2}{8l_B^2}},\end{aligned} \tag{3.7}$$

where $d_{00}(z_c, z_r), d_{10}(z_c, z_r), d_{01}(z_c, z_r)$ and $d_{11}(z_c, z_r)$ are holomorphic functions of z_r and z_c with well-defined parity in z_r. Specifically, antisymmetry dictates that Taylor

expansions of $d_{00}(z_c, z_r), d_{10}(z_c, z_r), d_{11}(z_c, z_r)$ only have odd powers in z_r whereas the Taylor expansion of $d_{01}(z_c, z_r)$ only have even powers in z_r. It will be advantageous to work with an orthogonal basis of two-particle states that preserves as far as possible a factorization into center-of-mass and relative parts. Note that unlike the lowest Landau level, higher Landau levels are *not* invariant subspaces of the relative or center-of mass angular momentum operators individually, hence unlike in the lowest Landau level, there are no good quantum numbers associated with these observables. This is related to the presence of the last term in Eq. (3.7). We thus write:

$$\begin{aligned}\psi(z_c, z_r) = \sum_{R,\ell} \big\{ & a_{R,\ell}\, \eta^r_{0,\ell}(z_r)\eta^c_{0,2R-\ell}(z_c) + \\ & b_{R,\ell}\, \eta^r_{0,\ell}(z_r)\eta^c_{1,2R-\ell}(z_c) + \\ & c_{R,\ell}\, \eta^r_{1,\ell}(z_r)\eta^c_{0,2R-\ell}(z_c) + \\ & d_{R,\ell}\, \left(\eta^r_{0,\ell}(z_r)\eta^c_{2,2R-\ell}(z_c) - \eta^r_{2,\ell-2}(z_r)\eta^c_{0,2R+2-\ell}(z_c)\right)/\sqrt{2} \big\},\end{aligned} \tag{3.8}$$

where functions $\eta^r_{k,m}(z_r)$ and $\eta^c_{k,m}(z_c)$ are obtained from $\eta_{k,m}(z)$ via substitutions $l_B \to \sqrt{2} l_B$ and $l_B \to l_B/\sqrt{2}$, respectively, ℓ is restricted to odd integers, and the $k = 0, 1$ Landau level wave functions were given above and those for $k = 2$ are also needed:

$$\eta_{2,m}(z) = \frac{(\bar{z}^2 z^{m+2} - 4l_B^2(m+2)\bar{z}z^{m+1} + 4l_B^4(m+2)(m+1)z^m)e^{-|z|^2/4l_B^2}}{\sqrt{2\pi 2^{m+5} l_B^{2m+10}(m+2)!}}. \tag{3.9}$$

It is easy to see that Eq. (3.8) reproduces the analytic structure of Eq. (3.7). Moreover, any $\psi(z_c, z_r)$, which we will always assume to be sufficiently rapidly decaying, can be expanded in the form Eq. (3.8), which follows from completeness properties of the η-symbols.

It is further easy to show that the Hamiltonian Eq. (3.1) is positive semidefinite for general n[1], which will be shown in particular for $n = 2$ below. Therefore, as in the more familiar $n = 1$, any zero energy states (zero modes) are exact ground states. One may further easily see that the familiar analyticity requirements for zero modes for

[1]For a rigorous proof concerning any Landau levels, see Appendix B.

$n = 1$ generalize as follows. For the two-particle state (3.8) *not* to be annihilated by H (i.e., to have any non-zero matrix elements within the image of P_n), its polynomial expansion (not including the Gaussian term) must have terms that are at most linear in both z_r and $\bar{z}_r$, viewed as independent variables[2]. With this in mind, working at fixed angular momentum $L_z = 2R$ at the moment, we see that all non-zero eigenstates of H must be contained in the six-dimensional sub-space spanned by the following states,

$$\eta^r_{1,-1}(z_r)\eta^c_{0,2R+1}(z_c)\,, \tag{3.10a}$$

$$\eta^r_{0,1}(z_r)\,\eta^c_{0,2R-1}(z_c)\,, \tag{3.10b}$$

$$\eta^r_{0,1}(z_r)\,\eta^c_{1,2R-1}(z_c)\,, \tag{3.10c}$$

$$\frac{\left(\eta^r_{0,1}(z_r)\,\eta^c_{2,2R-1}(z_c) - \eta^r_{2,-1}(z_r)\,\eta^c_{0,2R+1}(z_c)\right)}{\sqrt{2}}, \tag{3.10d}$$

$$\eta^r_{1,1}(z_r)\,\eta^c_{0,2R-1}(z_c)\,, \tag{3.10e}$$

$$\frac{\left(\eta^r_{0,3}(z_r)\,\eta^c_{2,2R-3}(z_c) - \eta^r_{2,1}(z_r)\,\eta^c_{0,2R-1}(z_c)\right)}{\sqrt{2}}\,. \tag{3.10f}$$

while its orthogonal complement (for given R) is spanned by states already annihilated

[2]This is proved in Appendix C.

by H. It follows from this that the Hamiltonian may be written in the form

$$H = \sum_R \sum_{i,j=1}^{6} m_{i,j} {Q_R^{(i)}}^\dagger Q_R^{(j)} \tag{3.11}$$

where the operators ${Q_R^{(i)}}^\dagger$, $i = 1, 2, 3, 4, 5, 6$, create the states in Eq. (3.10). Specifically, in second quantized form, these operators read:

$$Q_R^{(1)} = \frac{1}{2^{R+1/2}} \sum_{x=-R}^{R+1} \sqrt{\binom{2R+1}{R+x}} c_{1,R-x} c_{0,R+x}, \tag{3.12a}$$

$$Q_R^{(2)} = -\frac{1}{2^R} \sum_{x=-R}^{R} x \sqrt{\frac{1}{R}\binom{2R}{R+x}} c_{0,R-x} c_{0,R+x}, \tag{3.12b}$$

$$Q_R^{(3)} = \frac{1}{2^{R+1/2}} \sum_{x=-R}^{R+1} (1-2x) \sqrt{\frac{1}{2R+1}\binom{2R+1}{R+x}} \times c_{1,R-x} c_{0,R+x}, \tag{3.12c}$$

$$Q_R^{(4)} = -\frac{1}{2^{R+1/2}} \sum_{x=-R-1}^{R+1} x \sqrt{\frac{1}{2R+2}\binom{2R+2}{R+1+x}} \times c_{1,R-x} c_{1,R+x}, \tag{3.12d}$$

$$Q_R^{(5)} = \frac{1}{2^R} \sum_{x=-R}^{R+1} \left(2x^2 - 2x - R\right) \sqrt{\frac{1}{2R\,(2R+1)}\binom{2R+1}{R+x}} \times c_{1,R-x} c_{0,R+x}, \tag{3.12e}$$

$$Q_R^{(6)} = -\frac{1}{2^R\sqrt{3}} \sum_{x=-R-1}^{R+1} (2x^3 - (3R+2)x) \times \sqrt{\frac{1}{2R(2R+1)(2R+2)}\binom{2R+2}{R+1+x}} \times c_{1,R-x} c_{1,R+x}. \tag{3.12f}$$

As before, x is summed over (half)integer when R is an (half)integer. Possible values

for $R \pm x$ is non-negative for Landau level index $n = 0$, and are greater than or equal to -1 for $n = 1$, to accommodate for the $L_z = -1$ angular momentum state in the first excited Landau level. One may check that these operators satisfy $\langle 0|Q_R^{(n)} Q_{R'}^{(m)\dagger}|0\rangle = \delta_{n,m}\delta_{R,R'}$, as expected from the orthonormality of first quantized wave functions used in this analysis. The matrix elements m_{ij} in Eq. (3.11) turn out to be independent of R, and can be read of the following expression:

$$\begin{aligned} H = & \frac{1}{4\pi}\sum_R Q_R^{(1)\dagger} Q_R^{(1)} + \frac{3}{8\pi}\sum_R Q_R^{(4)\dagger} Q_R^{(4)} + \frac{1}{4\pi}\sum_R (Q_R^{(1)\dagger} Q_R^{(4)} + \text{h.c.}) \\ & + \frac{1}{4\pi}\sum_R Q_R^{(3)\dagger} Q_R^{(3)} \\ & + \frac{1}{4\pi}\sum_R Q_R^{(2)\dagger} Q_R^{(2)} + \frac{1}{2\pi}\sum_R Q_R^{(5)\dagger} Q_R^{(5)} + \frac{3}{8\pi}\sum_R Q_R^{(6)\dagger} Q_R^{(6)} \\ & - \frac{\sqrt{2}}{4\pi}\sum_R (Q_R^{(2)\dagger} Q_R^{(5)} + \text{h.c.}) - \frac{\sqrt{6}}{8\pi}\sum_R (Q_R^{(2)\dagger} Q_R^{(6)} + \text{h.c.}) + \frac{\sqrt{3}}{4\pi}\sum_R (Q_R^{(5)\dagger} Q_R^{(6)} + \text{h.c.}). \end{aligned} \tag{3.13}$$

It turns out that only four of the six eigenvalues of the m-matrix are non-zero: $(5+\sqrt{17})/(16\pi)$, $(5-\sqrt{17})/(16\pi)$, $1/(4\pi)$, and $9/(8\pi)$. Eigenstates corresponding to these non-zero eigenvalues are:

$$\frac{\sqrt{2}}{2\sqrt{17-\sqrt{17}}}\left((-1+\sqrt{17})Q_R^{(1)\dagger} + 4Q_R^{(4)\dagger}\right)|0\rangle, \tag{3.14a}$$

$$\frac{\sqrt{2}}{2\sqrt{17+\sqrt{17}}}\left((-1-\sqrt{17})Q_R^{(1)\dagger} + 4Q_R^{(4)\dagger}\right)|0\rangle, \tag{3.14b}$$

.

$$Q_R^{(3)\dagger}|0\rangle, \tag{3.14c}$$

$$\frac{1}{3}\left(-\sqrt{2}Q_R^{(2)^\dagger} + 2Q_R^{(5)^\dagger} + \sqrt{3}Q_R^{(6)^\dagger}\right)|0\rangle. \tag{3.14d}$$

If we denote them by $\tilde{Q}_R^{(1)\dagger}|0\rangle$, $\tilde{Q}_R^{(4)\dagger}|0\rangle$, $\tilde{Q}_R^{(3)\dagger}|0\rangle$ and $\tilde{Q}_R^{(2)\dagger}|0\rangle$, then the Hamiltonian can be written in diagonal form:

$$\begin{aligned} H = & \frac{5+\sqrt{17}}{16\pi}\sum_R \tilde{Q}_R^{(1)\dagger}\tilde{Q}_R^{(1)} \\ & + \frac{5-\sqrt{17}}{16\pi}\sum_R \tilde{Q}_R^{(4)\dagger}\tilde{Q}_R^{(4)} \\ & + \frac{1}{4\pi}\sum_R \tilde{Q}_R^{(3)\dagger}\tilde{Q}_R^{(3)} \\ & + \frac{9}{8\pi}\sum_R \tilde{Q}_R^{(2)\dagger}\tilde{Q}_R^{(2)}. \end{aligned} \tag{3.15}$$

The positive semi-definiteness of the Hamiltonian (3.15) leads to the conclusion that any zero mode of the Hamiltonian (3.15) must be a simultaneous zero energy eigenstate of each positive term $\tilde{Q}_R^{(k)\dagger}\tilde{Q}_R^{(k)}$, and, to this end, must be annihilated by each individual operator $\tilde{Q}_R^{(k)}$. Therefore any zero modes $|\psi_0\rangle$ must obey the zero mode condition:

$$\tilde{Q}_R^{(i)}|\psi_0\rangle = 0 \tag{3.16}$$

for $i = 1, 2, 3, 4$ and for any integer or half integer R. Equivalently, this condition on zero modes indicates that zero modes are annihilated by $Q_R^{(1)}$, $Q_R^{(4)}$, $Q_R^{(3)}$ and $\tilde{Q}_R^{(2)}$ since $Q_R^{(1)}$ and $Q_R^{(4)}$ are linear combinations of $\tilde{Q}_R^{(1)}$ and $\tilde{Q}_R^{(4)}$. So zero mode conditions can be reformulated as:

$$Q_R^{(1)}|\psi_0\rangle = 0, \tag{3.17a}$$

$$Q_R^{(3)}|\psi_0\rangle = 0, \tag{3.17b}$$

$$Q_R^{(4)}|\psi_0\rangle = 0, \tag{3.17c}$$

$$\tilde{Q}_R^{(2)}|\psi_0\rangle = 0 \tag{3.17d}$$

for any integer or half integer R. This generalizes the familiar statement for $n = 1$ Landau level, where the V_1 Haldane pseudopotential is a two-body projection operator onto states of relative angular momentum 1. Presently, for $n = 2$, and for given pair angular momentum $2R$, the Trugman-Kivelson interaction is the sum of four two-particle projection operators, each of which is associated with a one dimensional subspace spanned by $\tilde{Q}_R^{(i)\dagger}|0\rangle$, $i = 1, 2, 3, 4$. Note that it is no longer possible to ascribe definite relative angular momentum quantum numbers to these states, as relative angular momentum is not a good quantum number in higher Landau levels.

3.2 Derivation of general properties of root patterns on disk

With second quantized form of the parent Hamiltonian, we are now in a position to analyze properties of what we will call general root patterns of zero modes of this Hamiltonian. To this end, we will utilize a recently developed method[62] to extract root patterns of zero modes directly from the parent Hamiltonian, without any need for studying presupposed wave functions. This has the advantage that since rules for root patterns are arrived at directly as properties of the Hamiltonian, these rules immediately provide rigorous constraints on the zero mode counting for the respective Hamiltonian. In particular, upper bounds for the number of zero modes are immediately available (which we will subsequently show to be saturated), and in particular claims about the unprojected Jain state as the unique densest zero mode of its parent Hamiltonian are immediately established (and in some geometries, refined).

Such claims have appeared earlier in the literature[74], but, by our reading, have so far been based on numerics, and were thus limited to finite particle number. The present treatment will be free of such limitations.

We begin by clarifying what we mean by a root pattern. The notion of a root pattern has mainly appeared in the literature in the context of single component states, where root patterns are essentially simple product states associated to more complicated quantum Hall trial wave functions. The present situation involves Landau level mixing and is more akin to that in multi-component states, which is more complicated and was described in Refs. [83, 84, 4, 24]. We first remind the reader of what has been termed a "non-expandable" basis state[62] in the expansion of a zero mode in terms of orbital occupancy number basis states,

$$|\psi_0\rangle = \sum_{\{n\}} C_{\{n\}} |\{n\}\rangle \,. \tag{3.18}$$

Here, each $|\{n\}\rangle$ is a basis state created by a product of single particle creation operators $c^\dagger_{i,m}$. We will call a basis state $|\{n\}\rangle$ in this expansion non-expandable if it enters the expansion with non-zero coefficient $C_{\{n\}}$ and if it cannot be obtained from any other such basis state $|\{n'\}\rangle$, also with $C_{\{n'\}} \neq 0$, through "inward-squeezing" processes[11]. That is,

$$|\{n\}\rangle \neq c^\dagger_{l_1,j} c^\dagger_{l_2,i} c_{l_3,i-x} c_{l_4,j+x} \ldots |\{n'\}\rangle \,, \tag{3.19}$$

where a single inward squeezing process is a center-of-mass conserving inward pair hopping satisfying $i - x < i \leq j < j + x$, the l_1, l_2, l_3 and l_4 are *arbitrary* Landau level indices (thus generalizing the standard notion of inward squeezing for single Landau level one-component states), and the dots represent a multiplicative string of any finite number of such inward squeezing terms.

The existence of non-expandable states in any occupancy number spectral decomposition of the form (3.18) follows from the finiteness of the number of states available at given angular momentum. (We may of course limit the discussion to zero modes

of well-defined angular momentum without loss of generality). It turns out, as we will show below for the present case, that such non-expandable states are subject to certain very restrictive rules, which, in the context of single component single LL states, are known as generalized Pauli principles [62]. Occupancy number eigenstates satisfying these rules are generally known as root patterns. The above implies that every zero mode contains at least one root pattern in its orbital occupancy number spectral decomposition. Typically, a clever basis of zero modes may be chosen in a manner that there is precisely one such root pattern per zero mode. It then follows from the above that every $|\{n\}\rangle$ appearing in the zero mode's decomposition (3.18) may be obtained from its unique root pattern through inward squeezing processes. This then establishes a one-to-one correspondence between root patterns and zero modes. It is worth pointing out that while this correspondence has been established for a large class of single component quantum Hall states[11, 12], this is usually done by analysis of special analytic clustering conditions attributed to first-quantized zero mode wave functions. It may be less clear what the latter even means in the presence of Landau level mixing. Related to this, while root patterns are always characterized by definite *orbital* occupancies, it is only for single component states that this ensures root patterns to be simple non-entangled product states. In contrast, for the case of multi-component states, it was shown[83, 84] that additional rules require entanglement in the additional internal degree of freedom for root patterns that are (locally) at maximum possible filling fraction. A similar phenomenon will be observed here, where root patterns are found to retain some entanglement in the Landau level degrees of freedom. That is, we will define root patterns as states of definite single particle angular momentum occupancy numbers $n_j = \sum_l c^\dagger_{l,j} c_{l,j}$ satisfying certain rules given below, where moreover the Landau level degrees of freedom are subject to additional rules and may be required to be entangled in a certain way. The non-expandable part of the expansion (3.18) of a zero mode $|\psi_0\rangle$ must then always be a superposition of root patterns. We note that a first quantized wave function analysis was used in Refs. [83, 84] to deal with similar complications in the presence of spin degrees of freedom, using thin cylinder techniques. Here we find that the most efficient and

general approach to studying the structure of root patterns, as defined above, is to forgo first quantized wave functions, and work with the second quantized form of the zero mode condition as in Eq. (3.16). We find this particularly true in problems where degrees of freedom beyond pure guiding centers are present, e.g. spin and/or Landau level degrees of freedom. To this end we generalize the method introduced in Ref. [62] for single Landau level, single component states to states with an additional Landau level degrees of freedom.

In the following we will write second quantized wave function in terms of a string of numbers: x0!10..., where ! stands for an occupied orbital in the lowest Landau level (LLL), 1 represents an occupied orbital in the first excited Landau level (1st LL), x represents a particle in either LLL or 1st LL and 0 stands for an unoccupied orbital. Here orbitals are arranged in the order of ascending angular momenta stating with -1. Before proceeding to our main results, we will prove a few lemmas:

Lemma 1 There is no 101 in root patterns of the zero mode.

Proof. We will use the method of contradiction and the property that any root pattern is, by definition, non-expandable. Now let us assume that a root pattern $|\{n_{\text{root}}\}\rangle$ contains the string 101 in which 0 has angular momentum j. Then $|\{n_{\text{root}}\}\rangle$ can be written as

$$|\{n_{\text{root}}\}\rangle = c^\dagger_{1,j+1} c^\dagger_{1,j-1} |\{n'\}\rangle.$$

For $|x| > 1$, $c^\dagger_{1,j+x} c^\dagger_{1,j-x} |\{n'\}\rangle$ should have zero coefficient in the spectral decomposition of $|\psi_0\rangle$, i.e.,

$$\langle\{n'\}| c_{1,j-x} c_{1,j+x} |\psi_0\rangle = 0 \text{ for } |x| > 1,$$

otherwise $|\{n_{\text{root}}\}\rangle$ would be expandable. Thus, keeping only the $x = \pm 1$ terms,

$$\langle\{n'\}|Q^{(4)}_j|\psi_0\rangle = -2^{1/2-j}\sqrt{\binom{2j+2}{j+2}/(2j+2)}\langle\{n_{\text{root}}\}|\psi_0\rangle, \tag{3.20}$$

which is non-zero. This, however, contradicts the zero mode condition Eq. (3.17c). Thus, 101 must be excluded from any root pattern. ■

Using precisely the same logic, and the respectively appropriate zero mode condi-

tion, we may further obtain the following 2 lemmas:

Lemma 2 There is no 11 in root patterns of the zero mode.

Lemma 3 Two particles cannot occupy both LLL and 1st LL orbitals with the same angular momentum in root patterns of the zero mode.

Furthermore, we also have the following lemmas:

Lemma 4 There is no xx in root patterns of the zero mode.

Proof. According to *Lemma 2*, there is no 11 in the root pattern, so possible configurations of xx are !!, !1 and 1!. Thus we can write

$$|\psi_0\rangle = (\gamma_{0,0}c^\dagger_{0,j}c^\dagger_{0,j+1} + \gamma_{0,1}c^\dagger_{0,j}c^\dagger_{1,j+1} + \gamma_{1,0}c^\dagger_{0,j+1}c^\dagger_{1,j})|\{n'\}\rangle + \text{other terms}$$

if there is xx in the root pattern. Eq.(3.17a) and Eq.(3.17b) lead to $\sqrt{j+1}\gamma_{0,1} + \sqrt{j+2}\gamma_{1,0} = 0$ and $-\sqrt{j+1}\gamma_{0,1} + \sqrt{j+2}\gamma_{1,0} = 0$, respectively. Thus both $\gamma_{0,1}$ and $\gamma_{1,0}$ are zero. We then use Eq.(3.17d) to find that $\gamma_{0,0}$ is also zero. This concludes the proof. ■

Lemma 5 If x0x appears in root patterns, then proportions of coefficients of !0!, !01, and 10! are $2 : \sqrt{j+2} : -\sqrt{j}$, where j is the angular momentum of the unoccupied orbital in x0x.

Proof. we can write

$$\begin{aligned}|\psi_0\rangle =&(\alpha_{0,0}c^\dagger_{0,j-1}c^\dagger_{0,j+1} + \alpha_{0,1}c^\dagger_{0,j-1}c^\dagger_{1,j+1} + \alpha_{1,0}c^\dagger_{1,j-1}c^\dagger_{0,j+1}\\ &+ \beta_{0,1}c^\dagger_{0,j}c^\dagger_{1,j})|\{n'\}\rangle + \text{other terms}\end{aligned}$$

if there is x0x in root patterns. In the latter expression, the first three terms define the root pattern, whereas the fourth term is inward squeezed from the root pattern. Note that 101 is absent in root patterns because of *Lemma 1.* Using Eq.(3.17a), Eq.(3.17b) and Eq.(3.17d) in a manner analogous to the proofs of the preceding lemmas, we find that

$$\alpha_{1,0} = -\alpha_{0,1}\frac{\sqrt{j}}{\sqrt{j+2}}\ , \tag{3.21}$$

$$\beta_{0,1} = -2\alpha_{0,1}\frac{\sqrt{j}}{\sqrt{j+2}} \tag{3.22}$$

and

$$\alpha_{0,1} = \alpha_{0,0}\frac{\sqrt{j+2}}{2}. \tag{3.23}$$

Therefore proportions of coefficients of !0!, !01 and 10! are $2 : \sqrt{j+2} : -\sqrt{j}$. ■

Lemma 6 There is no x0x0x in root patterns of the zero mode.

Proof. From the first four lemmas, the only allowed x0x0x in root patterns are 10!01, 10!0!, !010!, !0!01 and !0!0!. If we assume that the angular momentum of the first orbital in above patterns is j, then from *Lemma 5.*, proportions of coefficients of 10!0!, 10!01 and 1010! are $2 : \sqrt{j+4} : -\sqrt{j+2}$. 1010! is excluded from root patterns by virtue of *Lemma 1*, therefore 10!0! and 10!01 are also excluded. Using the same trick, remaining three possible configurations are excluded form root patterns as well.

■

Lemma 7 If x00x is in root patterns, then there is no generic constraint on proportions of coefficients of !00!, !001, and 100! and 1001.

Lemma 7 is listed here for completeness, as together with the remaining lemmas, it gives a complete set of rules for the construction of root patterns in one-to-one correspondence with the zero modes of the Hamiltonian. That all the root patterns allowed by these rules do indeed correspond to a zero mode they dominate follows only from explicit construction of such zero modes, as will be discussed below. The constraints imposed by Lemmas 1-6, on the other hand, can then be used to rigorously imply that the set of zero modes thus constructed is complete. It may be instructive, though, to see why the logic used to derive Lemmas 1-6 does not give additional constraints in the situation relevant to Lemma 7. To briefly show on this, we may write

$$\begin{aligned}|\psi_0\rangle =&(ac^\dagger_{0,j}c^\dagger_{0,j+3} + bc^\dagger_{0,j}c^\dagger_{1,j+3} + dc^\dagger_{1,j}c^\dagger_{0,j+3} + ec^\dagger_{1,j}c^\dagger_{1,j+3}\\ &+ fc^\dagger_{0,j+1}c^\dagger_{0,j+2} + gc^\dagger_{0,j+1}c^\dagger_{1,j+2} + hc^\dagger_{1,j+1}c^\dagger_{0,j+2} + ic^\dagger_{1,j+1}c^\dagger_{1,j+2})\\ &\times |\{n'\}\rangle + \text{other terms}\end{aligned}$$

if x00x is part of a root pattern. Lemma 7 is then related to the fact that there are eight unknown coefficients and four zero mode conditions (2.33).

Using Lemmas 1-6, we are able to arrive at the following important theorem about the zero mode of the Hamiltonian:

Theorem 1 At given particle number N and given angular momentum L, the number of linearly independent zero modes of the Hamiltonian (3.1) is no greater than the number of dominance patterns consistent with lemmas 1-6.

Proof. Assume that the number of linearly independent zero modes is greater than the number of root patterns consistent with lemmas 1-6. Then it is possible to make a non-trivial linear combination of such zero modes that is orthogonal to all root patterns consistent with these lemmas. Any non-expandable states present in the spectral decomposition (3.18) would therefore violate these lemmas, which is a contradiction, as any non-trivial superposition of linearly independent zero modes is still a zero mode. ■

As a result, we immediately have the following

Corollary 1.1 For given particle number N, there exist no zero modes of the Hamiltonian (3.1) (disk geometry) at angular momentum $L < L_e(N) := 5/4N^2 - 2N$ for N even, and at angular momentum $L < L_o(N) := 5/4(N-1)^2 + 1/2(N-3)$ for N odd. In the latter case, if a zero mode exists at $L = L_o(N)$, it is unique, whereas for N even, a zero mode at $L = L_e(N)$ can be at most doubly degenerate.

Proof. The densest possible root patterns consistent with lemmas 1-6 are, respectively, 100x0x00x0x...00x0x for N odd, and 100x0x00x0x...00x0x001, 100x0x00x0x...00x0x00! for N even, where "densest" means in particular that no consistent root patterns exist at angular momenta less than the ones corresponding to these patterns, which can be seen to be $L_e(N)$ for even N and $L_o(N)$ for odd N. Hence, any zero mode with angular momentum less than these bounds would have non-expandable basis states in their spectral decomposition that violate some of lemmas 1-6, which is not possible. The statement about uniqueness/degeneracy of zero modes with L at the threshold $L_{e/o}(N)$ then follows from Theorem 1. ■

For any zero mode, let l_{max} be the highest angular momentum of the single particle orbitals that are at least partially occupied in that zero mode, i.e., that have $\langle \sum_i c^\dagger_{i,l} c_{i,l} \rangle \neq 0$. Then we finally have

Corollary 1.2 Any zero mode of the Hamiltonian (3.1) (disk geometry) has $l_{\mathsf{max}} \geq 5(N-1)/2-1$ for N odd, and $l_{\mathsf{max}} \geq 5N/2-3$ for N even. Any zero modes satisfying these bounds have angular momentum $L_o(N)$ or $L_e(N)$, respectively, and in particular the statements about degeneracy from Corollary 1.1 apply.

Proof. Same as Corollary 1.1, and using the definition of "non-expandable". In particular, any basis state present in the spectral decomposition is non-expandable or can be obtained by inward squeezing from a non-expandable state, hence statements about l_{max} are obtained from knowledge of non-expandable states. ■

If we define the filling factor ν of a zero mode as N/l_{max}, then Corollary 1.2 implies that the densest (highest) filling factor for which zero modes exist is bounded from above by 2/5 in the thermodynamic limit. This bound is, of course, saturated, as the corresponding wave function is known[37, 74]. In the following, we will describe the relation between admissible root patterns and the analytic structure of corresponding full zero mode wave functions.

3.3 Zero mode counting and generators on disk

As argued in the introduction, a good quantum Hall parent Hamiltonian will have zero energy eigenstates that not only characterize the incompressible quantum fluid sufficiently uniquely, but also encode the proper edge theory of the system. The rules derived in the preceding section thus far only suggest a certain zero mode structure, but, with the exception of (the yet unproven) lemma 7, only constrain this structure without guaranteeing the existence of any zero modes. It is, however, worth noting that all of this was derived from the second quantized operators $Q_R^{(i)}$ alone, and, if we took lemma 7 for granted, the entire zero mode structure in terms of root patterns would follow correctly from this analysis. To prove lemma 7 and thus the existence of all the zero modes of Eq. (3.1) with $n = 2$, we briefly make contact with first quantized presentation of zero modes, though at least in part we will see below that an operator-based approach is also possible. (In all aspects, such an operator-based

approach has been constructed previously for the $n = 1$ case related to the 1/3-Laughlin state, and in fact for all the Laughlin states[62, 22, 55]. We will comment more on the situation below.)

The analysis of Appendix C implies that a sufficient (and necessary) property of any zero mode is that the associated analytic many-body wave function contains the factor $(z_i - z_j)^2$ for all i, j. This is, in fact, a quite special property of the cases $n = 1$ and $n = 2$ in Eq. (3.1). More generally, zero modes of Eq. (3.1) may be linear combinations of terms containing the factors $(z_i - z_j)^2$, $(z_i - z_j)(\bar{z}_i - \bar{z}_j)$, and $(\bar{z}_i - \bar{z}_j)^2$, which, by symmetry, must be true for all i, j. That is, a zero mode vanishes at least to second order in the separation of any pair of coordinates. For $n \leq 2$, however, the third term is prohibited by Landau-level projection, and the second then always necessitates another factor of $z_i - z_j$ by anti-symmetry, such that the first term still covers all possible cases for having a second order zero. This renders the $n = 2$ of Eq. (3.1) rather special. While the presence of the first excited Landau level allows terms in $\bar{z}_i$ to be present in the wave function, the zero mode condition can thus be stated only in terms of the holomorphic variables z_i. Indeed, it is only for $n \leq 2$ that the ground state of Eq. (3.1) is in the Jain sequence of states.

Thanks to the work done in the preceding section, for now it will do to note that divisibility of the wave function by $\psi_{1/2} = \prod_{i<j}(z_i - z_j)^2$, the bosonic $\nu = 1/2$ Laughlin-Jastrow factor, is a sufficient criterion for a wave function to be a zero mode. The necessity of this criterion (for $n = 2$), i.e., the completeness of the resulting zero mode space, will then soon follow from Theorem 1. We thus consider zero mode wave functions of the form $\psi_{1/2} p(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N)$, where p is an arbitrary polynomial of the requisite anti-symmetry and at most first order in the $\bar{z}_i$ (so as for $\psi_{1/2}\, p$ to be in the first two Landau levels), and we drop the obligatory Gaussian factor for simplicity. It is clear that a suitable basis for these polynomials is given by $S_{\{\mathfrak{n}\}}(z_1, \bar{z}_1, \ldots)$, where $S_{\{\mathfrak{n}\}}$ is a Slater determinant of single particle states in the lowest and the first excited Landau level, with occupancies determined by a set of occupancy numbers $\{\mathfrak{n}\}$. Hence

we wish to study zero modes of the form

$$\psi_{1/2}(z_1, \dots) S_{\{\mathfrak{n}\}}(z_1, \bar{z}_1, \dots)\,. \tag{3.24}$$

We note that zero modes of this form are naturally viewed as composite fermion (CF) states, where any fermion forms a composite object with two bosonic flux quanta. In particular, if the CF-occupancy configuration $\{\mathfrak{n}\}$ is chosen to represent two equally filled Landau levels, one recovers the Jain 2/5 state, and one easily verifies that this state saturates the bounds of the Corollaries of the last section. Therefore, the Jain 2/5 state is the densest zero mode of Eq. (3.1) for $n = 2$, unique up to the twofold degeneracy mentioned in Corollary 1.1.

We emphasize that while notationally similar to the *electron* occupancy numbers $\{n\}$ labeling basis states in Eq. (3.18), the labels $\{\mathfrak{n}\}$ represent *composite fermion* occupancy numbers and must be well distinguished from the labels $\{n\}$. To analyze the root patterns underlying the zero modes (3.24), we make use of well known rules[13] for products of polynomials with known root patterns, generalized to the case where non-holomorphic variables (or more than a single Landau level) are present. Every CF-occupancy configuration $\{\mathfrak{n}\}$ is naturally the dominance or root pattern, in the sense defined above, of associated wave function $S_{\{\mathfrak{n}\}}(z_1, \bar{z}_1, \dots)$, as it is the only Slater determinant appearing in this wave function. We may represent such a CF-occupancy pattern using strings ("words") made up of characters X, 0, 1, and !, where the last three have the analogous meaning as in our notation for root patterns of full zero mode wave functions (but refer to CFs), and X now means a double occupancy of the associated angular momentum state in both Landau levels. As before, the first character can only be 1 or 0, see Fig.3.1. Moreover, as is well known[72], the bosonic Laughlin factor $\psi_{1/2}$ is dominated by the pattern !0!0!0!0 .... Patterns may generally be associated to partitions $l_N + l_{N-1} + \ldots + l_1 = L$, where $l_i \geq l_{i+1}$ is the angular momentum of the ith particle in the pattern, and L is the total angular momentum of the pattern. It is well known[13] that when wave functions dominated by patterns with partitions $\{l_i\}$ and $\{l'_i\}$ are multiplied, the resulting wave function

is dominated by a pattern with partition $\{l_i + l'_i\}$. It is easy to see that these rules, when applied to the present situation, imply that the multiplication of $\psi_{1/2}$ by the Slater-determinant $S_{\{\mathfrak{n}\}}$ leads to a wave function with a root pattern obtained from that associated to as follows. The character ! is replaced !00, (! $\rightarrow$!00, rule 1). An X in the CF-pattern associated to $S_{\{\mathfrak{n}\}}$ corresponds to the case where $l_i = l_{i+1}$ in the resulting partition, signifying two particles with identical angular momenta but different Landau level indices. The resulting ambiguity in ordering these two particles leads to the situation described as x0x in the root pattern of the resulting zero mode, i.e., we have the rule $X \rightarrow$x0x00 (rule 2). That the underlying configurations !0!, 10!, and !01 indeed occur with the ratios claimed by lemma 5 could be verified directly from Eq. (3.24), but this is not necessary, since Eq. (3.24) is definitely a zero mode, and then the proof of lemma 5 applies. A "1" in the CF-pattern associated to $S_{\{\mathfrak{n}\}}$ leads to at least two non-expandable Slater determinants in Eq. (3.24), one obtained from the replacement $1 \rightarrow 100$ (rule 3.a), and one from $1 \rightarrow$!00 (rule 3.b). Eq. (3.24) is then dominated by more than a single root pattern. However, it is clear that if we ignore rule 3.b for the moment, rules 1-3.a establish a one-to-one correspondence (see Fig.3.1) between CF-occupation number patterns $\{\mathfrak{n}\}$ of N particles occupying orbitals with angular momentum up to l_{max} and root pattern of N particles occupying orbitals with angular momentum up to $l_{\mathsf{max}}+2(N-1)$ (where the addition of $2(N-1)$ can be thought of as being due to flux attachment.) Let us now denote all possible root pattern states by $|p\rangle$ where p labels the pattern, and let us choose an ordering of these patterns such that the number of 1s in the pattern increases monotonously for patterns associated to the same partition $\{l_i\}$. Furthermore, we may order patterns associated to different partitions according to increasing $S(\{l_i\}) := \sum_i l_i^2$. Finally, let us order the CF-occupancy patterns $\{\mathfrak{n}\}$ in the same way, by means of the one-to-one correspondence. We then see that the matrix

$$C_{p,\{\mathfrak{n}\}} = \langle p|\psi_{1/2} S_{\{\mathfrak{n}\}}\rangle \tag{3.25}$$

is upper triangular with non-zero diagonal and thus invertible. This implies that for

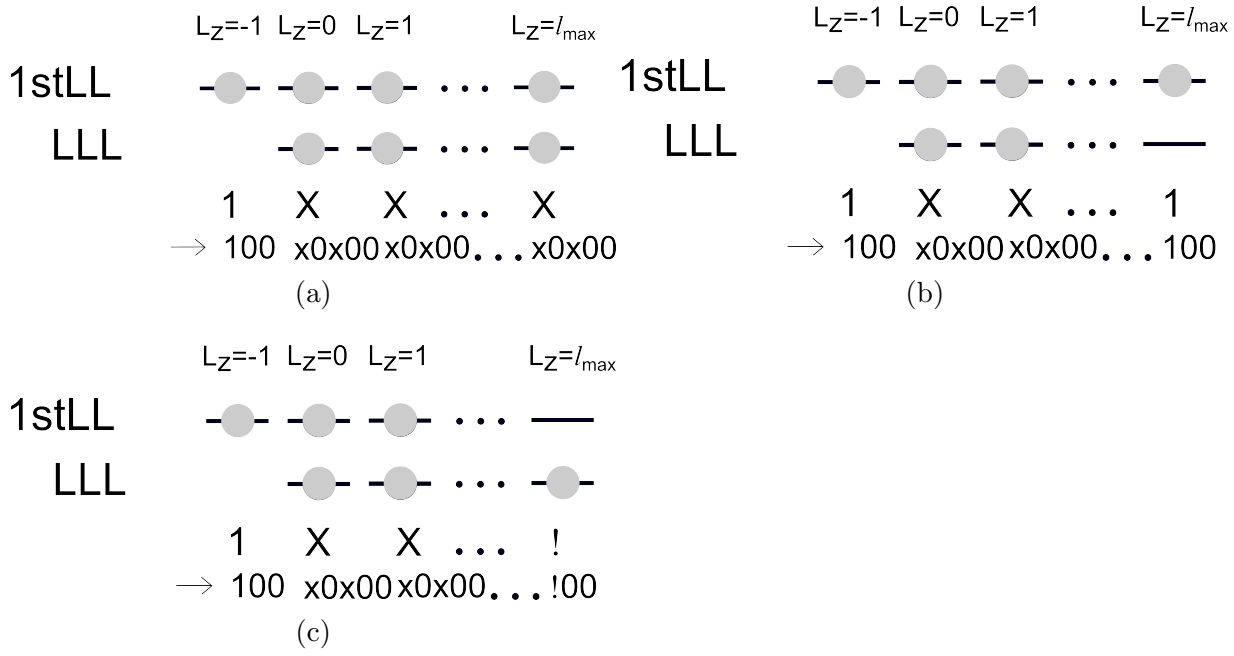

Figure 3.1: One-to-one correspondence between CF-occupancy patterns of N particles with the largest angular momentum of orbitals being l_{max} and root patterns of N particles with the largest angular momentum of orbitals being $l_{\mathsf{max}} + 2(N-1)$ using rule 1, 2 and 3.a.

each root pattern satisfying lemmas 1-6, there is a superposition of zero modes of the form (3.24) that is dominated precisely by this root pattern, with no other of the state $|p\rangle$ present in its spectral decomposition (3.18). This establishes thus the one-to-one correspondence between zero modes and root patterns satisfying lemmas 1-6.

We will now discuss zero mode generators. Zero mode generators are defined in the sense that when acting on known zero modes, they give new zero modes with larger angular momentum. Such generators pertaining to one single Landau level has been found[62, 22, 55]. Here zero mode generators involving two Landau levels are

$$P_d^{(1)} = \sum_r \sqrt{\frac{(r+d)!}{(r+1)!}} c^\dagger_{0,r+d} c_{1,r} \tag{3.26}$$

and

$$P_d^{(2)} = \sum_r \Big(\sqrt{\frac{(r+d)!}{r!}} c^\dagger_{0,r+d} c_{0,r} + \sqrt{\frac{(r+d+1)!}{(r+1)!}} c^\dagger_{1,r+d} c_{1,r} \Big). \tag{3.27}$$

It is easy to see that both $P_d^{(1)}|\psi_0\rangle$ and $P_d^{(2)}|\psi_0\rangle$ obey the zero mode condition (2.33) by virtue of following commutators:

$$[Q_R^{(1)}, P_d^{(1)}] = 0, \tag{3.28a}$$

$$[Q_R^{(3)}, P_d^{(1)}] = 2^{1-d/2}\sqrt{\frac{(2R)!}{(2R-d+1)!}}Q_{R-d/2}^{(4)}, \tag{3.28b}$$

$$[Q_R^{(4)}, P_d^{(1)}] = 0, \tag{3.28c}$$

$$\begin{aligned}[\tilde{Q}_R^{(2)}, P_d^{(1)}] = &-\frac{2^{(3-d)/2}}{3}\sqrt{\frac{(2R-1)!}{(2R-d+1)!}}\Big((d-1)Q_{R-d/2}^{(1)} \\ &+\sqrt{2R-d+1}Q_{R-d/2}^{(3)} + 2(d-1)Q_{R-d/2}^{(4)}\Big),\end{aligned} \tag{3.28d}$$

$$[Q_R^{(1)}, P_d^{(2)}] = 2^{1-d/2}\sqrt{\frac{(2R+1)!}{(2R-d+1)!}}Q_{R-d/2}^{(1)}, \tag{3.28e}$$

$$[Q_R^{(3)}, P_d^{(2)}] = 2^{1-d/2}\sqrt{\frac{(2R)!}{(2R-d)!}}Q_{R-d/2}^{(3)}, \tag{3.28f}$$

$$[Q_R^{(4)}, P_d^{(2)}] = 2^{1-d/2}\sqrt{\frac{(2R+1)!}{(2R-d+1)!}}Q_{R-d/2}^{(4)}, \tag{3.28g}$$

$$\begin{aligned}[\tilde{Q}_R^{(2)}, P_d^{(2)}] =&2^{(1-d)/2}\sqrt{\frac{(2R-1)!}{(2R-d+1)!}}\Big(\frac{2d(d-1)}{3}Q_{R-d/2}^{(1)} \\ &+ d(d-1)Q_{R-d/2}^{(4)} \\ &+\sqrt{2(2R-d)(2R-d+1)}\tilde{Q}_{R-d/2}^{(2)}\Big).\end{aligned} \tag{3.28h}$$

We can also find that zero mode generators commute with each other:

$$[P_d^{(i)}, P_{d'}^{(j)}] = 0, \tag{3.29}$$

with $i, j = 1, 2$.

Now with these zero mode generators, we are able to obtain the zero mode counting which gives 1,2,5... for $\Delta L_z = 0, 1, 2....$ Specifically, if we want to get a new zero mode with the total angular momentum increased by 1,we can act either $P_1^{(1)}$ or $P_1^{(2)}$ on old zero modes. Likewise, there are 5 zero mode operators which can increase angular momentum by 2. They are $P_2^{(1)}, P_2^{(2)}, P_1^{(1)}P_1^{(2)}, (P_1^{(1)})^2$ and $(P_1^{(2)})^2$. The result of zero mode counting exactly agrees with the edge theory of Jain state, which involves two U(1) chiral bosons[106].

3.4 Projection of Trugman-Kivelson interaction to the first excited Landau level

We have projected the Trugman-Kivelson interaction onto the lowest two Landau levels and have studied properties of its zero modes. A question naturally arises that what will happen if we project the Trugman-Kivelson interaction to the first excited Landau level only. It is difficult to answer this question in the first quantized approach. However, the second quantized approach naturally provides an easy way to do that. All we have to do is to retain the terms in Eq. 3.13 involving all particles in the first excited Landau level. This result in

$$H = \frac{3}{8\pi} \sum_R Q_R^{(4)\dagger} Q_R^{(4)} + \frac{3}{8\pi} \sum_R Q_R^{(6)\dagger} Q_R^{(6)}, \tag{3.30}$$

which are the second quantized form of the Haldane V_1 and V_3 pseudopotentiala in the first excited Landau level. Consequently, the filling factor of the highest density zero mode of the Hamiltonian in Eq. 3.30 is 1/5 in the first excited Landau level, which is

far from obvious had we not chosen the second quantization approach. We conjecture that if we project the Trugman-Kivelson interaction onto the nth Landau level, then the highest filling factor of zero modes would be $1/(3+2n)$. This hypothesis will be investigated in our future work.

3.5 Conclusion

In this chapter, we have analyzed general zero mode properties of the parent Hamiltonian of the unprojected Jain 2/5 state. We observed that the ground state degeneracy varies according to the number of particles in the disk geometry. We have established zero mode counting in terms of root patterns. Preference was given to the second quantized methods, where zero mode properties are derived not from given analytic wave functions, but from a "lattice" Hamiltonian in Eq. (3.13) and associated zero mode conditions (3.17). The second quantized form of the parent Hamiltonian was extensively discussed in the planar geometry, and was presented as a positive linear combination of four two-particle projection operators for a given pair of angular momentum as seen in Eq. 3.15. We also identified two independent second quantized generators of zero modes, only one of which has a simple interpretation in the first quantization.

Chapter 4

Summary of the dissertation

In Chapter 2 and Chapter 3 of this dissertation, we have studied the zero modes of Haldane pseudopotential and Trugman-Kivelson interaction projected to certain Landau levels purely from the approach of second quantization. Although there have already been studies of zero modes of both pseudopotential from the approach of first quantization or in numerics, the second quantized approach provides a complimentary viewpoint to that of the first quantized one. Moreover, in second quantization the above Hamiltonians assume the form of positive semi-definite frustration 1D lattice Hamiltonians after the degree of freedom associated with dynamic momenta are projected out, leaving the guiding center as the effective degree of freedom. It turns out that the guiding center degree of freedom is essential to the fractional quantum Hall effect. The aforementioned 1D lattice Hamiltonian is usually of the form

$$H = \sum_k \sum_R Q_R^{(k)\dagger} Q_R^{(k)}\,, \tag{4.1}$$

where R is half of the total angular momentum. The ground eigenstates of this Hamiltonian have zero eigenenergies, hence they are called zero modes of the Hamiltonian. These zero modes must satisfy zero mode conditions in that they are annihilated by $Q_R^{(k)}$ with all possible values of k and R.

In Chapter 2, we have studied zero modes of such frustration free lattice Hamil-

tonian as Haldane pseudopotential. We constructed its zero modes at filling factor $1/M$ iteratively from $N-1$ to N particles and proved they indeed satisfy zero mode conditions. We also proved the uniqueness of the zero mode at $\nu = 1/M$. All these proofs are based on the algebraic properties of edge excitation operators and those operators pertaining to the Hamiltonian. we have also obtained all zero modes at $\nu < 1/M$ using edge excitation operators,.

In Chapter 3, we have projected Trugman-Kivelson interaction, which is known in numerics to be the Hamiltonian stabilizing unprojected Jain 2/5 state on sphere, to two lowest Landau levels on disk and obtained a semi-definite frustration 1D lattice Hamiltonian in the second quantized approach. We found two independent zero mode generators whose forms are hard to guess in first quantized approach, and also found that, contrary to the fact that the zero mode is always non degenerate in spherical geometry, on disk this Hamiltonian has the unprojected Jain state and another state as zero modes at $\nu = 2/5$ if the particle number is even.

Appendices

Appendix A

The Origin of Landau Levels.

To study the effect of the magnetic field on the energy spectrum of a single electron in the disk geometry, we begin with the Hamiltonian for a single electron of mass m in 2D(2 dimensions) in a magnetic field $\mathbf{B} = (0, 0, -B)$ perpendicular to that 2D surface:

$$H = \frac{\Pi_x^2}{2m} + \frac{\Pi_y^2}{2m}. \tag{A.1}$$

where dynamical momenta operators

$$\Pi_x = p_x + eA_x \tag{A.2}$$

and

$$\Pi_y = p_y + eA_y. \tag{A.3}$$

We can choose a symmetric gauge pertaining to the disk geometry $\mathbf{A} = \frac{B}{2}(y, -x, 0)$ where x and y are Cartesian coordinates on disk. If we transform Cartesian coordinates into polar coordinates r and φ, then the Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{L_z^2}{2mr^2} - \frac{eBL_z}{2m} + \frac{e^2B^2r^2}{8m}, \tag{A.4}$$

where the angular momentum operator of the electron

$$L_z = -i\hbar\frac{\partial}{\partial\varphi} \tag{A.5}$$

.The wave function of the electron must then obey the Schrödinger equation:

$$H\psi = E\psi. \tag{A.6}$$

We note that the Hamiltonian H commutes with the angular momentum operator L_z:

$$[H, L_z] = 0, \tag{A.7}$$

hence the energy eigenstate ψ is also the eigenstate of the angular momentum operator. Consequently, we can write $\langle \mathbf{r}|\psi\rangle$ as $f(r)e^{ij\varphi}$ with the integer j being the angular momentum of the single electron divided by $\hbar$. Thus the Schrödinger equation can be simplified as

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{j^2\hbar^2}{2mr^2}-\frac{eBj\hbar}{2m}+\frac{e^2B^2r^2}{8m}\right)f(r) = Ef(r). \tag{A.8}$$

The energy E is found to be quantized as

$$E = \hbar\omega\left(s+\frac{|j|}{2}-\frac{j}{2}+\frac{1}{2}\right) \tag{A.9}$$

which are called Landau levels(LL), where the cyclotron frequency ω is equal to eB/m and s is an integer. The radial part of the wave function is

$$f(r) = r^{|j|}L_s^{|j|}\left(\frac{r^2}{2l_B^2}\right)e^{-\frac{r^2}{4l_B^2}}, \tag{A.10}$$

where the magnetic length $l_B = \sqrt{\hbar/eB}$. and the generalized Laguerre polynomial

$$L_s^{|j|}(x) = \sum_{i=0}^{s}(-1)^i\begin{pmatrix} s+|j| \\ s-i \end{pmatrix}\frac{x^i}{i!}.$$

The normalized wave function with the energy eigenvalue $\hbar\omega\left(s+\frac{|j|}{2}-\frac{j}{2}+\frac{1}{2}\right)$ and angular momentum eigenvalue $j\hbar$ is then

$$\psi_{s,j}(\mathbf{r})=\frac{(-1)^s\sqrt{s!}}{\sqrt{2\pi 2^{|j|}l_B^{2|j|+2}(s+|j|)!}}r^{|j|}e^{ij\varphi}L_s^{|j|}\left(\frac{r^2}{2l_B^2}\right)e^{-\frac{r^2}{4l_B^2}}. \tag{A.11}$$

If $j\geqslant 0$, the energy $E=\hbar\omega\left(s+\frac{1}{2}\right)$, else $E=\hbar\omega\left(s-j+\frac{1}{2}\right)$. Thus the wave function corresponding to the angular momentum $j<0$ and the energy $\hbar\omega\left(n+\frac{1}{2}\right)$ is

$$\psi_{n+j,j}(\mathbf{r})=\frac{(-1)^{n+j}\sqrt{(n+j)!}}{\sqrt{2\pi 2^{-j}l_B^{-2j+2}n!}}\bar{z}^{-j}\sum_{k=0}^{n+j}(-1)^k\binom{n}{n+j-k}\frac{(\bar{z}z/(2l_B^2))^k}{k!}e^{-\frac{r^2}{4l_B^2}}. \tag{A.12}$$

where $z=x+i\ y=r\ e^{i\varphi}$ is complex coordinate on the disk and j should satisfy $-n\leq j<0$. After introducing a new variable $i=k-j,\ \psi_{n+j,j}(\mathbf{r})$ can be written as

$$\frac{(-1)^n\sqrt{n!}}{\sqrt{2\pi 2^j l_B^{2j+2}(n+j)!}}z^j\sum_{i=-j}^{n}(-1)^i\binom{n+j}{n-i}\frac{(\bar{z}z/(2l_B^2))^i}{i!}e^{-\frac{r^2}{4l_B^2}}. \tag{A.13}$$

So the wave function corresponding to the energy $\hbar\omega\left(n+\frac{1}{2}\right)$ and the angular momentum $j\hbar$ can been written in the same form:

$$\eta_{n,j}(\mathbf{r})=\frac{(-1)^n\sqrt{n!}}{\sqrt{2\pi 2^j l_B^{2j+2}(n+j)!}}z^j\sum_{i=0}^{n}(-1)^i\binom{n+j}{n-i}\frac{(\bar{z}z/(2l_B^2))^i}{i!}e^{-\frac{r^2}{4l_B^2}}. \tag{A.14}$$

where $j\geq -n$.

The wave function of the electron in first three Landau levels used in Chapter 3 are

$$\eta_{0,j}(z)=\frac{z^j e^{-|z|^2/4l_B^2}}{\sqrt{2\pi 2^j l_B^{2j+2}j!}}, \tag{A.15}$$

$$\eta_{1,j}(z)=\frac{(\bar{z}z^{j+1}-2l_B^2(j+1)z^j)\,e^{-|z|^2/4l_B^2}}{\sqrt{2\pi 2^{j+2}l_B^{2j+6}(j+1)!}}, \tag{A.16}$$

and

$$\eta_{2,j}(z) = \frac{(\bar{z}^2 z^{j+2} - 4l_B^2(j+2)\bar{z}z^{j+1} + 4l_B^4(j+2)(j+1)z^j)e^{-|z|^2/4l_B^2}}{\sqrt{2\pi 2^{j+5} l_B^{2j+10}(j+2)!}}. \tag{A.17}$$

It turns out that the number of electrons per area in each Landau level in the thermodynamic limit is eB/h [54]. Resultantly, the macroscopic degeneracy per Landau level is equal to $\frac{\Phi}{h/e}$, the number of magnetic flux quanta penetrating the 2D surface since the elementary flux quantum Φ_0 is just h/e. The filling factor of Landau levels ν can be defined as the electron density n divided by the number of electrons per area in each Landau level eB/h, so the filling factor

$$\nu = \frac{nh}{eB}. \tag{A.18}$$

The same problem can also be solved using two sets of ladder operators[54]. We can define creation and annihilation operators associated with dynamic momenta operators:

$$a^\dagger = \frac{-il_B}{\sqrt{2}\hbar}(\Pi_x - i\Pi_y) \tag{A.19}$$

and

$$a = \frac{il_B}{\sqrt{2}\hbar}(\Pi_x + i\Pi_y) \tag{A.20}$$

with the relation $[a, a^\dagger] = 1$. The Hamiltonian A.1 can be written as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{A.21}$$

So again this Hamiltonian will have the energy spectra $\hbar\omega\left(n + \frac{1}{2}\right)$ with n being an nonnegative integer. There are also guiding center operators which commute with dynamic momenta operators:

$$X = x + \frac{\Pi_y}{m\omega} \tag{A.22}$$

and

$$Y = y - \frac{\Pi_x}{m\omega} \tag{A.23}$$

with $[\Pi_x, X] = [\Pi_x, Y] = [\Pi_y, X] = [\Pi_y, Y] = 0$. We can define ladder operators pertaining to guiding center operators:

$$b = \frac{1}{\sqrt{2}l_B}(X - iY) \tag{A.24}$$

and

$$b^\dagger = \frac{1}{\sqrt{2}l_B}(X + iY) \tag{A.25}$$

with $[b, b^\dagger] = 1$.

Therefore, we can have the simultaneous eigenstate of H and $b^\dagger b$: $|n, m\rangle$ for which

$$H|n, m\rangle = \hbar\omega(n + \frac{1}{2})|n, m\rangle \tag{A.26}$$

and

$$b^\dagger b|n, m\rangle = m|n, m\rangle \tag{A.27}$$

with $n, m \geq 0$. In the symmetric gauge $\mathbf{A} = \frac{B}{2}(y, -x, 0)$, the angular momentum L_z is equal to $\hbar(b^\dagger b - a^\dagger a)$. Consequently, $|n, m\rangle$ has the angular momentum $l_z = \hbar(m - n) \geq -n\hbar$.

Appendix B

The Positive Semi-definiteness of Trugman-Kivelson Interaction.

To show that the two-body Trugman-Kivelson interaction is positive semi-definiteness, we must prove that its diagonal matrix elements in the basis of its eigenstates are always nonnegative. To this end, we can calculate its diagonal matrix elements between any of its two-body eigenstate which can be written as $f(\mathbf{r_1}, \mathbf{r_2})$:

$$\int dx_1 dy_1 dx_2 dy_2 \bar{f}(\mathbf{r_1}, \mathbf{r_2}) \nabla^2_{\mathbf{r_1}} \delta(x_1 - x_2)\delta(y_1 - y_2) f(\mathbf{r_1}, \mathbf{r_2}), \tag{B.1}$$

which, after integration by parts, can be simplified to

$$\int dx_2 dy_2 dx_1 dy_1 \delta(x_1 - x_2)\delta(y_1 - y_2)\nabla^2_{\mathbf{r_1}}(\bar{f}f). \tag{B.2}$$

Note that $\nabla^2_{\mathbf{r_1}}(\bar{f}f) = f\nabla^2_{\mathbf{r_1}}\bar{f} + \bar{f}\nabla^2_{\mathbf{r_1}}f + 2\nabla_{\mathbf{r_1}}f \cdot \nabla_{\mathbf{r_1}}\bar{f}$. Since we are considering fermions throughout Chapter 3, $f(\mathbf{r_1}, \mathbf{r_2})$ and $\bar{f}(\mathbf{r_1}, \mathbf{r_2})$ will vanish if we let $x_1 = x_2$ and $y_1 = y_2$. So the remaining non-vanishing term is

$$2\int dx_2 dy_2 dx_1 dy_1 \delta(x_1 - x_2)\delta(y_1 - y_2)\nabla_{\mathbf{r_1}} f(\mathbf{r_1}, \mathbf{r_2}) \cdot \overline{\nabla_{\mathbf{r_1}} f(\mathbf{r_1}, \mathbf{r_2})}, \tag{B.3}$$

which is nonnegative.

Appendix C

The Conditions Under Which Trugman-Kivelson Interaction Has Non-zero Matrix Elements.

We begin with the matrix elements of Trugman-Kivelson interaction between two general wave functions $f(\mathbf{r_1}, \mathbf{r_2})$ and $g(\mathbf{r_1}, \mathbf{r_2})$ describing two fermions of coordinates $\mathbf{r_1}$ and $\mathbf{r_2}$, respectively:

$$\int dx_1 dy_1 dx_2 dy_2 \bar{f}(\mathbf{r_1}, \mathbf{r_2}) \nabla^2_{\mathbf{r_1}} \delta(x_1 - x_2) \delta(y_1 - y_2) g(\mathbf{r_1}, \mathbf{r_2}). \tag{C.1}$$

$f(\mathbf{r_1}, \mathbf{r_2})$ can be expanded in $z_r = z_1 - z_2$ and its complex conjugate:

$$f(\mathbf{r_1}, \mathbf{r_2}) = \sum_{i,j} z_r^i \bar{z}_r^j d_{ij}(z_c, \bar{z}_c). \tag{C.2}$$

with i and j being nonnegative integers. The same applies to $g(\mathbf{r_1}, \mathbf{r_2})$:

$$g(\mathbf{r_1}, \mathbf{r_2}) = \sum_{k,l} z_r^k \bar{z}_r^l e_{ij}(z_c, \bar{z}_c). \tag{C.3}$$

with k and l being nonnegative integers. Therefore the expansion of $\bar{f}(\mathbf{r_1}, \mathbf{r_2}) g(\mathbf{r_1}, \mathbf{r_2})$ contains such general term as $(z_1 - z_2)^m (\bar{z}_1 - \bar{z}_2)^n h(z_1, \bar{z}_1, z_2, \bar{z}_2)$ where $m = j + k$

and $n = i + l$. Hence we will consider the integration

$$\int dx_1 dy_1 dx_2 dy_2 \nabla^2_{\mathbf{r_1}} \delta(x_1 - x_2)\delta(y_1 - y_2)(z_1 - z_2)^m(\bar{z}_1 - \bar{z}_2)^n h\left(z_1, \bar{z}_1, z_2, \bar{z}_2\right). \quad \text{(C.4)}$$

Upon integration by parts, it can be simplified to

$$\begin{aligned} 2\int dz_2 d\bar{z}_2 dz_1 d\bar{z}_1 \delta(z_1 - z_2)\delta(\bar{z}_1 - \bar{z}_2)\Big(& mn(z_1 - z_2)^{m-1}(\bar{z}_1 - \bar{z}_2)^{n-1} h\left(z_1, \bar{z}_1, z_2, \bar{z}_2\right) \\ &+ m(z_1 - z_2)^{m-1}(\bar{z}_1 - \bar{z}_2)^n \frac{\partial h\left(z_1, \bar{z}_1, z_2, \bar{z}_2\right)}{\partial \bar{z}_1} \\ &+ n(z_1 - z_2)^m(\bar{z}_1 - \bar{z}_2)^{n-1} \frac{\partial h\left(z_1, \bar{z}_1, z_2, \bar{z}_2\right)}{\partial z_1} \\ &+ (z_1 - z_2)^m(\bar{z}_1 - \bar{z}_2)^n \frac{\partial^2 h\left(z_1, \bar{z}_1, z_2, \bar{z}_2\right)}{\partial z_1 \partial \bar{z}_1}\Big). \end{aligned} \quad \text{(C.5)}$$

It is easy to see that the matrix element will vanish if any of m or n is greater than or equal to 2. For the matrix elements to be non-zero, both m and n must be less than or equal to 1. It is easy to see that the only two sets of solutions are

$$i = 1, j = 0, k = 1, l = 0, \quad \text{(C.6)}$$

and

$$i = 0, j = 1, k = 0, l = 1. \quad \text{(C.7)}$$

Consequently, to have any non-zero matrix elements of Trugman-Kivelson interaction, the expansion of fermionic wave function must have terms that are at most linear in z_r or in $\bar{z}_r$.

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